2. CONVOLUTION

Convolution sum. Response of d.t. LTI systems at a certain input signal

Any signal multiplied by the unit impulse
= the unit impulse weighted by the value of the signal in 0:

\[ x[n] \cdot \delta[n] = x[0] \cdot \delta[n] \]

Delayed unit impulse:

\[ x[n] \cdot \delta[n-k] = x[k] \cdot \delta[n-k] \]

Any signal can be "broken" into a sum of shifted and weighted unit impulses

\[ x[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k] \]
• The impulse response: response to $\delta[n]$

$\delta[n] \quad S_d \quad h[n]$

LTI system

$\delta[n-k] \quad S_d \quad h[n-k]$

LTI system
Convolution sum

\[ y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k] = x[n] \ast h[n] = h[n] \ast x[n] \]

Example 1)

\[(x \ast y)[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k] \]

two signals with finite durations \(N_1\) and \(N_2\)

Convolution - duration of \(N_1 + N_2 - 1\).
Causal Discrete-Time Signals

If the input signal and the system are causal then the output signal is also causal.

\[ x[n] = 0 \quad \text{and} \quad h[n] = 0, \quad n < 0 \quad \Rightarrow \quad y[n] = \sum_{k=0}^{n} x[k] h[n-k] \]

If the system causal:

\[ h[n] = 0, \quad n < 0 \quad \Leftrightarrow \quad h[n] = h[n] \cdot \sigma[n], \quad \forall n \in \mathbb{Z} \]

\[ y[n] = \sum_{k=0}^{\infty} x[n-k] h[k] = \sum_{k=-\infty}^{n} x[k] h[n-k] \]
BIBO stability

• “bounded input bounded output” (BIBO) stable system
• BIBO stability condition
A discrete–time LTI system is stable if and only if its impulse response is absolutely summable

\[ \sum_{k=-\infty}^{\infty} |h[k]| < \infty, h[n] \in l^1 \]

The stability of the accumulator

• \( h[n] \) - the unit step \( \sigma[n] \), bounded signal
• finite difference eq. (x[n] causal) :

\[ y[n] = x[n] \ast \delta[n] \Rightarrow y[n] = \sum_{k=0}^{n} x[k] \]
• The corresponding output signal:

\[ y[n] = \sum_{k=0}^{n} 1 = 1 + 1 + \ldots + 1 = n + 1 \]

• The output signal is not bounded.
• The accumulator is unstable.
• Used in practice,
  – limited interval of summation \( n \), or,
  – from time to time, the output set on zero.
Properties of convolution
Neutral element

- unit impulse $\delta[n]$ is neutral element.
  $x[n] \ast \delta[n] = x[n]$, for any signal $x[n]$

\[ \delta[n] \rightarrow h[n] \rightarrow h[n] \]

$h[n]$ - the system’s impulse response.

Identity system

- impulse response $\delta[n]$
- input $\delta[n] \rightarrow$ output $\delta[n]$. 

\[ \delta[n] \rightarrow h[n] \rightarrow h[n] \rightarrow \delta[n] \]
Delay system

Impulse response \( h[n] = \delta[n-n_0] \).

Response: a time shifted variant of the input signal

\[
x[n] * \delta[n-n_0] = x[n-n_0]
\]

Associativity. Series interconnection of 2 d.t. LTI systems

- two LTI systems \( h_1[n], h_2[n] \) connected in series

\[
\begin{align*}
y_1[n] &= x[n] * h_1[n] \\
y[n] &= y_1[n] * h_2[n] = (x * h_1)[n] * h_2[n] \\
y[n] &= x[n] * (h_1 * h_2)[n]
\end{align*}
\]
Associativity

- The convolution is **associative**.
  \[(x * h_1)[n] * h_2[n] = x[n] * (h_1[n] * h_2[n])\]

- The response of the equivalent system at the unit impulse signal
  \[h[n] = h_1[n] * h_2[n]\]

Stability for series interconnection of two 2 d.t. LTI systems

- By the cascade connection of **two stable systems** another **stable system** is obtained.

\[h_1[n] \in l^1, \ h_2[n] \in l^1 \Rightarrow (h_1 * h_2)[n] \in l^1\]
Commutativity

- The convolution sum is **commutative**.
- The equivalent system is
  \[ h[n] = h_1[n] * h_2[n] = h_2[n] * h_1[n] \]
- their order is **not important** in the cascade connection!!!
The Inverse System

Two systems connected in series
The output of the second system - the original input signal
\[ h[n] * h_i[n] = \delta[n] \]

Inverse system - example

- delay system:
  \[ h[n] = \delta[n - n_0] \]
- inverse system of a delay system
  \[ h_i[n] = \delta[n + n_0] \]
- time shift in the other sense of the time axis.
- not a causal system
Distributivity.
Parallel interconnection of dt LTI systems

• convolution is distributive with respect to addition

\[(x * h_1)[n] + (x * h_2)[n] = (x * (h_1 + h_2))[n]\]

Proof

\[y[n] = y_1[n] + y_2[n] = (x * h_1)[n] + (x * h_2)[n] =\]
\[= \sum_{k=-\infty}^{\infty} x[k] \cdot h_1[n-k] + \sum_{k=-\infty}^{\infty} x[k] \cdot h_2[n-k] =\]
\[= \sum_{k=-\infty}^{\infty} x[k] \cdot (h_1[n-k] + h_2[n-k]) =\]
\[= (x * (h_1 + h_2))[n]\]
Response of a d.t. LTI system at the unit step. The unit step response

- The **unit step response** $s[n]$.

$$x[n] = \sigma[n],$$
$$y[n] = s[n] = h[n]*\sigma[n] = \sum_{k=-\infty}^{n} h[k]$$

$$h[n] = s[n] - s[n-1]$$

- For causal systems:

$$h[n] \equiv 0, \text{ for } n < 0 \Rightarrow s[n] = \sum_{k=0}^{n} h[k]$$

- For an accumulator the unit step response a ramp signal (see previous slides).

$$x[n] = \sigma[n] \Rightarrow s[n] = n + 1$$
Finite Impulse (FIR) and Infinite Impulse (IIR) Response d.t. Systems

- There are two types of impulse responses
  - with finite duration (FIR) and
  - with infinite duration (IIR).

\[ h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases} \]
**FIR**

- The finite difference equation
  \[ \sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \]

- a FIR system has the property:
  \[ a_0 \neq 0 \text{ and } a_1 = a_2 = \ldots = a_N = 0 \]

**FIR**

- the output signal using the finite diff.eq.:
  \[ y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k] \]

- the output signal using convolution:
  \[ y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k] \]
FIR

- Identification:

\[ h[k] = \begin{cases} 
\frac{b_k}{a_0}, & k = 0, 1, \ldots, M \\
0, & \text{in rest}
\end{cases} \]

- The impulse response has **finite duration** \( M+1 \) – hence the name “**finite impulse response system**”

IIR systems

- If the last proposition is not verified, for example if

\[ a_1 \neq 0 \text{ and } a_0 \neq 0 \]

- then we obtain a IIR system.
IIR systems - example

\[ y[n] - 0.5y[n-1] = x[n] \]

Initial condition
\[ y[-1]=0 \]

\[ x[n] = \delta[n] \Rightarrow y[n] = h[n] \]

- \( n=0 \Rightarrow h[0]=1 \)
- \( n=1 \Rightarrow h[1]=0.5 \)
- \( n=2 \Rightarrow h[2]=0.5^2 \)
- \( n=3 \Rightarrow h[3]=0.5^3 \), and so on

\[ h[n] = 0.5^n \sigma[n] \]

-infinite duration

Implementation of
d.t. LTI systems
described by finite differences
equations with constant coefficients
Implementation

• D.t. LTI systems are mathematically described by finite difference equations with constant coefficients.
• Implemented using standard sub-systems:
  – delay systems,
  – multipliers with constant,
  – adders.

\[
\begin{align*}
  a_0 y[n] + a_1 y[n-1] &= b_0 x[n] + b_1 x[n-1] \\
  z[n] &= a_0 y[n] + a_1 y[n-1] \\
\end{align*}
\]

• a delay sub-system: input signal \(x[n]\) into \(x[n-1]\)
• two multipliers with constants \(b_0\) and \(b_1\): \(b_0 x[n]\) and \(b_1 x[n-1]\)
• one adder

Implemented using the system a)
Direct form I

\[ y[n] = \frac{1}{a_0} (z[n] - a_1 y[n-1]) \]

Moving average part

\[ a_0 y[n] + a_1 y[n-1] = b_0 x[n] + b_1 x[n-1] \]

Autoregressive part

Subsystem 1

Subsystem 2

MA

AR
\[ N^{th} \text{ order system } \sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] ; \]
\[ z[n] = \sum_{k=0}^{M} b_k x[n-k] \]
\[ y[n] = \frac{1}{a_0} \left( z[n] - \sum_{k=1}^{N} a_k y[n-k] \right) \]

- For a FIR discrete-time system:
  \[ y[n] = \frac{1}{a_0} (z[n]) \]
- sub-system 2 = multiplier with constant \(1/a_0\)
FIR system - transversal form

• substitute the $M$ adders from subsystem 1 with a single adder with $M$ inputs

• we have supposed that $a_0 = 1$.

Direct form II

• For 1st order LTI system direct form I: series interconnection of two sub-systems, 1 and 2.
• in the series interconnection, the systems’ order is not important
• Reverse order and have an equivalent form of the direct form I: series interconnection of the sub-system 2 with the sub-system 1.
Direct form II

- Systems’ order is not important (series interconnection)
- Reverse order - series connection of subsystem 2 with subsystem 1
- Equivalent form of the direct form I: direct form II

Remove redundant delay system

- two times the same signal $v[n-1]$ - system a).
- So, a delay subsystem is redundant.
- Remove the delay subsystem $\Rightarrow$ implementation b).
- **direct form II** of a 1st order LTI d.t. system.
- **minimum number of subsystems** (delay systems, multipliers and adders).
Direct form II for $N$’th order system

- We have supposed that $M=N$
- **If $M>N$** then we can suppose that
  \[ a_{N+1}, a_{N+2}, \ldots, a_M = 0 \]
  and we can use the same implementation.
- **If $M<N$** then we can suppose that
  \[ b_{M+1}, b_{N+2}, \ldots, b_N = 0 \]
  and we can use the same implementation.
The convolution product.
The response of c.t. LTI systems at a specified input signal

Convolution product. C.t. signals

- C.t. LTI system, mathematically described by the operator $S$.
- find the output signal $y(t)$ when the input signal $x(t)$ is known.

\[ x(t) \xrightarrow{S} y(t) \]

LTI system
Convolution product. C.t. signals

• **Reminder (chp.1):** Filtering property of the Dirac unit impulse, $\delta(t)$

$$\int_{-\infty}^{\infty} \varphi(\tau) \delta(\tau) d\tau = \varphi(0)$$

• Function test $x(\tau)$, and as impulse the shifted version with $t$ (time is $\tau$)

$$\int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau = x(t)$$

$\mathbf{S}$

LTI system

• the output signal

$$y(t) = S\{x(t)\}$$

$$= S\left\{\int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau \right\}$$

$$= \int_{-\infty}^{\infty} x(\tau) S\{\delta(t-\tau)\} d\tau$$
Convolution product. Definition

\[ y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) \, d\tau \]

- convolution product of continuous-time signals \( x \) and \( h \).
- compute the response of a given system, with known impulse response (\( h \)) to a known input signal (\( x \)).

Neutral element

- **Neutral element** for convolution: Dirac distribution, \( \delta(t) \).

\[ x(t) * \delta(t) = x(t), \text{ for any signal } x(t) \]
Commutativity

- With the notation: \( t - u = \tau \), we have:
  \[
  (f \ast g)(t) = \int_{-\infty}^{\infty} g(u)f(t-u)\,du = (g \ast f)(t)
  \]

- So, the convolution product is **commutative** almost everywhere.

Some remarks

- If the input signal \( x(t) \in L^1 \) and \( h(t) \in L^1 \) (the system is stable)
- Then the convolution exists and the output signal
  \[
  y(t) = x(t) \ast h(t) \in L^1
  \]
Some remarks

- If the input signal $x(t) \in L^2$ and the impulse response system $h(t) \in L^2$
- then the convolution product $x(t) \ast h(t)$
  exists, is bounded and continuous

Some remarks

- If the input signal $x(t) \in L^2$ and the system is stable $h(t) \in L^1$
- Then the output signal has a finite energy
  $y(t) = x(t) \ast h(t) \in L^2$
Example 1)

\[ f(t) = \sigma(t) - \sigma(t - T_1) \]
\[ g(t) = \sigma(t) - \sigma(t - T_2) \]

length of the convolution’s support
\[ = T_1 + T_2 \] (sum of lengths of the supports of the two functions)

\[ (f * g)(t) = \begin{cases} 
0, & t < 0 \\
t, & 0 \leq t < T_2 \\
T_2, & T_2 \leq t \leq T_1 \\
T_1 + T_2 - t, & T_1 \leq t \leq T_1 + T_2 \\
0, & t > T_1 + T_2 
\end{cases} \]

- Denote \( g(-\tau) = z(\tau) \)
- Time-shifting of \( z(\tau) \) with \( t \) to the right
  \[ t < 0, \quad f(\tau)g(t - \tau) = 0; \]
  \[ 0 < t \leq T_2, \quad (f * g)(t) = \int_{0}^{t} d\tau = t; \]
  \[ T_2 \leq t \leq T_1, \quad (f * g)(t) = \int_{t-T_2}^{t} d\tau = T_2; \]
  \[ T_1 \leq t \leq T_1 + T_2, \quad (f * g)(t) = \int_{t-T_1}^{T_1} d\tau = T_1 + T_2 - t; \]
  \[ t > T_1 + T_2, \quad f(\tau)f(t - \tau) = 0, \quad (f * g)(t) = 0. \]
Example 2)

\[ f(t) = \sigma\left(t + \frac{T}{2}\right) - \sigma\left(t - \frac{T}{2}\right); \]
\[ g(t) = f(t); \]
\[ (f * g)(t) = T\left(1 - \frac{|t|}{T}\right)\left(\sigma(t + T) - \sigma(t - T)\right); \]

Example 3)

\[ f(t) = \frac{1}{\sqrt{|t|}} \frac{1}{|t| + 1}, \quad f(t) * f(t) = ? \]

- \( f(0) \to \infty; f(t) \) and \( |f(t)| \) are even.
- Function \( f \) belongs to \( L^1 \)
  \[ \int_{-\infty}^{\infty} |f(t)| dt = 2\int_{0}^{\infty} \frac{du}{u^2 + 1} = 2\arctan u \bigg|_{0}^{\infty} = 2 \cdot \frac{\pi}{2} = \pi < \infty \]
- \( f * f \) is convergent a.e.; but not for \( t=0 \)
  \[ (f * f)(0) = \int_{-\infty}^{\infty} f(t) \cdot f(-\tau) d\tau = \int_{-\infty}^{\infty} \frac{1}{|t| + 1} d\tau \]
Example 4)

\[ f(t) = \sigma\left(t + \frac{T}{2}\right) - \sigma\left(t - \frac{T}{2}\right); \]
\[ g(t) = a' \sigma(t), \ 0 < a < 1; \]
\[ (f \ast g) \in L^2 \text{ has infinite support.} \]

\[ (f \ast g)(t) = \frac{1}{\ln \left(\frac{1}{a}\right)} \left[\left(1 - a^{-\frac{T}{2}}\right) \sigma\left(t + \frac{T}{2}\right) - \left(1 - a^{-\frac{T}{2}}\right) \sigma\left(t - \frac{T}{2}\right)\right]. \]

- \( t < -T/2, f \ast g(t) = 0. \)
- \( t > -T/2 \) partial superposition - for \(-T/2 < t < T/2:\)

\[ (f \ast g)(t) = \int_{-T/2}^{t} a^\tau d\tau = \frac{1}{\ln \left(\frac{1}{a}\right)} \left( -a^{-\frac{T}{2}} + 1 \right). \]
- \( t > -T/2 \) complete superposition - for \( t \geq T/2: \)

\[ (f \ast g)(t) = \int_{-T/2}^{T/2} a^\tau d\tau = \frac{1}{\ln \left(\frac{1}{a}\right)} \left( a^{-\frac{T}{2}} - a^{\frac{T}{2}} \right). \]
Example 5) the integrator

- impulse response = $\sigma(t)$.
  
  \[ x(t) = \delta(t) \Rightarrow y(t) = h(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = \sigma(t) \]

- step response = ramp signal:
  \[
  (\sigma * \sigma)(t) = \int_{-\infty}^{\infty} \sigma(\tau)\sigma(t-\tau)d\tau = \begin{cases} 
  t, & t \geq 0 \\
  0, & t < 0
  \end{cases} = t \cdot \sigma(t).
  \]

- The result is not bounded, so the integrator is not stable.
- Despite this fact, the integrators can be used in practice, but only for finite duration input signals (case in which the output is bounded).
Associativity

- The convolution is associative
  \[(f(t) * g(t)) * h(t) = f(t) * (g(t) * h(t))\]
- If
  1. \(f, g, h \in L^1\);
  2. \(f, g, h \in L^1_{\text{loc}}\) and two of them have compact support;
  3. \(f, g, h \in L^1_{\text{loc}}\) and all three have like support a closed set included in \([0, \infty)\).

Unit step response of an LTI system

- response to the unit step signal.
  \[s(t) = h(t) * \sigma(t) = \int_{-\infty}^{t} h(\tau) d\tau\]
- Its derivative is the impulse response.
  \[s'(t) = h(t)\]
• response of the system $h$ to the ramp signal $x(t) = t\sigma(t) \Rightarrow y(t) = S\{t\sigma(t)\} = h*(t\sigma(t))$;
• The 2nd derivative of $y(t)$ is $h(t)$.
\[ y''(t) = h*(t\sigma(t))'' = h*\delta = h. \]

\[ x(t) = t\sigma(t) \quad \Rightarrow \quad y(t) = h(t)*t\sigma(t) \]
LTI system \[ y''(t) = h(t) \]

• find the impulse response $h(t)$ of a system
  – the derivation of its step response or
  – the double derivation of its response to a ramp signal.
The unit step response of a causal LTI system

- for a causal system, impulse response = zero for $t<0$ or $h(t) = h(t) \cdot \sigma(t)$

$$s(t) = h(t) \ast \sigma(t) = \int_0^t h(\tau) d\tau$$

- Causal input signal and system $\Rightarrow$ causal output

$$y(t) = \int_0^t x(t - \tau) h(\tau) d\tau$$

BIBO stability for continuous-time LTI systems

- “bounded input bounded output” (BIBO) stable system

- BIBO stability condition

A continuous–time LTI system is stable if and only if its impulse response is absolutely integrable

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty, h(t) \in L^1$$
An example – the integrator

\[ h(t) = \sigma(t) \in L^1; \quad y(t) = \int_{-\infty}^{t} x(\tau) d\tau \]

\[ y(t) = \int_{0}^{t} x(\tau) d\tau \] (input signal is causal)

Bounded input \( x(t) = \sigma(t) \) \( \Rightarrow \) unbounded output \( y(t) = \int_{0}^{t} 1 \cdot d\tau = t. \)

If \( x(t) \) has finite duration then \( y(t) \) is bounded.

• the integrator is not stable.
• It can be used in practice because all the practical signals have finite duration.

Practical significance of the convolution product’s properties

• Equivalent system for series interconnection of 2 systems has impulse response

\[ h(t) = h_2(t) \ast h_1(t) = h_1(t) \ast h_2(t) \]
Inverse system. Identity system

- Two systems connected in series
- Output – original input signal
  \[ h(t) * h_i(t) = \delta(t) \]
- The system connected in series with \( h(t) \) and \( h_i(t) \) is an identity system
  \[ y(t) = x(t) \]

Parallel interconnection of LTI systems

- Equivalent system for parallel interconnection of 2 systems has impulse response
  \[ h(t) = h_1(t) + h_2(t) \]
Implementation of c.t. LTI systems with linear differential equations & constant coefficients

Direct form II with differentiators.

\[
\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{N} b_k \frac{d^k x(t)}{dt^k} \quad a_N \neq 0
\]

- direct form II: delay systems (discrete-time) → differentiators (cont. time).
- difficult to construct differentiators in continuous-time
Direct form II with integrators.

- it is preferable to use integrators.
- Integrate N times a differential equation \( \Rightarrow \) an integral equation. With the notations
  \( y(0), y(1), \ldots, y(N), x(0), x(1), \ldots, x(N) \)
- we obtain the integral equation:

\[
\sum_{k=0}^{N} a_k y(N-k)(t) = \sum_{k=0}^{N} b_k x(N-k)(t)
\]

\[
\sum_{k=0}^{N} d^k y(t) = \sum_{k=0}^{N} b_k d^k x(t) \quad a_N \neq 0
\]

\[
y_{(0)}(t) = y(t),
\]

\[
y_{(1)}(t) = y(t) * \sigma(t) = \int_{-\infty}^{t} y(\tau_1) d\tau_1
\]

\[
y_{(2)}(t) = y(t) * \sigma(t) * \sigma(t) = \int_{-\infty}^{t} \int_{-\infty}^{\tau_1} y(\tau_1) d\tau_1 d\tau_2
\]

\[\ldots\]

\[
y_{(k)}(t) = y_{(k-1)}(t) * \sigma(t) = \int_{-\infty}^{t} \int_{-\infty}^{\tau_1} \int_{-\infty}^{\tau_2} \ldots \int_{-\infty}^{t} y(\tau_1) d\tau_1 d\tau_2 \ldots d\tau_{k-1} d\tau_k
\]

\[
x_{(0)}(t) = x(t) ; x_{(1)}(t) = x(t) * \sigma(t), \ldots
\]
Example for 2nd order c.t. system

\[ L \frac{d^2 y(t)}{dt^2} + R \frac{dy(t)}{dt} + C y(t) = x(t) \]

\[ \sum_{k=0}^{N} a_k y_{(n-k)}(t) = \sum_{k=0}^{N} b_k x_{(n-k)}(t), \]

\[ LC y_{(0)}(t) + RC y_{(1)}(t) + y_{(2)}(t) = x_{(2)}(t) \]
• identification:

\[ a_0 = 1 \; ; \; a_1 = RC \; ; \; a_2 = LC \; \; \; \; b_0 = 1 \]

• Using these coefficients we can write the corresponding integral equation
• we obtain the canonical form II with integrators.
Transversal structure for FIR systems

- continuous-time FIR systems have the impulse response:

\[ h(t) = \sum_{k=0}^{N} h_k \delta(t - kT). \]

- implemented using the transversal structure

\[
y(t) = x(t)h_0 + x(t-T)h_1 + \ldots + x(t-NT)h_N, \\
y(t) = x(t) * \sum_{k=0}^{N} h_k \delta(t - kT) = x(t) * h(t), \\
h(t) = \sum_{k=0}^{N} h_k \delta(t - kT).
\]