Modulation with continuous wave

• representations in time and frequency for two types of continuous wave modulation:
  – Amplitude modulation, AM -amplitude
  – Angle modulation,
    • frequency modulation (FM) - instantaneous frequency
    • phase modulation (PM) - instantaneous phase

• Purpose of a communication system:
  – transport a signal (a message) over a channel
  – deliver a reliable estimate to a user

• Example:
  – radio system: efficient in a freq. range > 30 kHz,
  – baseband signals = audio signals (0-20kHz)
  – Frequency shifting => modulation

• The message signal that contains information, generated by sources of information, is a baseband signal

• Modulation – information transfer from the modulating wave to carrier.
• Modulation / demodulation
  – (1) shifting frequency range of message signal into another one- suitable for transmission over the channel
  – (2) corresponding shift back to the original frequency range after reception of the signal
• Two most common used forms of carriers
  – Sinusoidal wave
  – Periodic pulse wave
• two main classes of modulation
  – Continuous wave (CW)
  – Pulse modulation

Amplitude modulation
• The amplitude of a carrier sine wave is modified according to a message signal= information

Angle modulation
• instantaneous frequency / phase of the carrier sine wave varies with the message
  – Frequency modulation
  – Phase modulation
Essential components of a communication system, using continuous-wave (CW) modulation

The noise from the channel decreases performance of the overall scheme

Amplitude modulation vs Angle modulation (exponential)
**Amplitude modulation**

Sinusoidal carrier wave: \( c(t) = A_c \cos \omega_c t \)

Modulating signal: \( x(t) \)

AM signal: \( s(t) = A_c \left[ 1 + k_a x(t) \right] \cos(\omega_c t) \).

\( k_a \left[ V^{-1} \right] \) - amplitude sensitivity of the modulator

**Modulation degree or percentage** (index):

\[ m = \left| k_a x(t) \right|_{\text{max}} \cdot 100 \% ; 0 < m \leq 1; m = k_a A_m \text{ (message=sine wave)} \]

\( f_M = \) maximum frequency of the modulating signal

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**varying percentage of modulation**

The amplitude of a harmonic signal is positive: \( A_c \left[ 1 + k_a x(t) \right] \geq 0 \)

\[ \Rightarrow \left| k_a x(t) \right| \leq 1 \ \forall t \]

If \( \left| k_a x(t) \right| > 1 \Rightarrow \text{overmodulation} \)

1) \( |k_a x(t)| \leq 1 \)

2) \( |k_a x(t)| > 1 \): overmodulation; **envelope distortions**, phase inversion in the carrier
Measuring modulation index

\[ m = \frac{A_{\text{max}} - A_{\text{min}}}{A_{\text{max}} + A_{\text{min}}} \times 100 \% \]

Valid when the modulating signal is a sine wave

**AM Spectrum**

\[ S(\omega) = \mathcal{F} \{ A_t \cos \omega_c t \} + \mathcal{F} \{ A_k x(t) \cos \omega_c t \} = \]
\[ = \pi A_t \left[ \delta(\omega - \omega_c) + \delta(\omega + \omega_c) \right] + \frac{1}{2\pi} A_k X(\omega) * \pi \left[ \delta(\omega - \omega_c) + \delta(\omega + \omega_c) \right]. \]

\[ S(\omega) = \pi A_t \left[ \delta(\omega - \omega_c) + \delta(\omega + \omega_c) \right] + \frac{k_A}{2} \left[ X(\omega - \omega_c) + X(\omega + \omega_c) \right]. \]

\[ S(f) = \frac{A_t}{2} \left[ \delta(f - f_c) + \delta(f + f_c) \right] + \frac{k_A}{2} \left[ X(f - f_c) + X(f + f_c) \right]. \]

The notation \( S(f) \) is used in communications.
Magnitude spectrum for the baseband signal and AM signal

\[ S(\omega) = \mu A_0 \left[ \delta(\omega - \omega_c) + \delta(\omega + \omega_c) \right] + \frac{k_m A_m}{2} \left[ X(\omega - \omega_c) + X(\omega + \omega_c) \right] \]

Condition to recover correctly the message signal & Bandwidth

- Upper and lower sidebands do not overlap if \( \omega_C - \omega_M > 0 \)

\[ f_c \gg f_M = B \text{ (message's bandwidth)} \]

- Bandwidth of the modulated signal, \( B_T \) is double of the bandwidth of the message (modulating) signal, \( B \)

\[ B_T = 2B \]
AM advantages and disadvantages

**simple implementation**
- used from the beginning in radio transmission
- cheaper

- bandwidth is 2x bandwidth of modulating wave

- **low energy efficiency**
- AM spectrum: the carrier ~ no information ⇒ waste of power
- Solution: suppress one of the sidebands and carrier ⇒ linear AM

**Modulator**

- nonlinear device, i.e. diode
\[ u_1(t) = A_c \cos \omega_c t + x(t) \Rightarrow u_2(t) = \left[ A_c \cos \omega_c t + x(t) \right] g(t) \]

\[ g(t) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos \left[ (2n-1) \omega_c t \right] \]

\[ u_2(t) \approx \frac{A_c}{2} \cos \omega_c t + \frac{A_c}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \left\{ \cos 2n\omega_c t + \cos \left[ (2n-2) \omega_c t \right] \right\} + \frac{x(t)}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} x(t) \cos \left[ (2n-1) \omega_c t \right] \]

For \( \omega_c \gg \omega_M \), in the neighborhood of \( \omega_c \):

\[ \frac{A_c}{2} \cos \omega_c t + \frac{2}{\pi} x(t) \cos \omega_c t \]

AM signal - separated by band-pass filtering centered on \( \omega_c \)

**Demodulator: envelope detector**

- Low-pass filtering of \( u_2(t) \) -> capacitor
- Removal of the DC component
Power of AM signal

\[ x(t) = A_m \cos \omega_m t \]
\[ s(t) = [A_c + K_u A_m \cos \omega_m t] \cos(\omega_c t) \]
\[ = A_c \cos(\omega_c t) + \frac{mA}{2} \cos[(\omega_c - \omega_m) t] + \frac{mA}{2} \cos[(\omega_c + \omega_m) t] \]

Power of the modulating signal \( P_m = \frac{A_m^2}{2} \)

Power of the AM signal : \( P_s = \frac{A_c^2}{2} + \frac{m^2 A_c^2}{8} + \frac{m^2 A_c^2}{8} = \frac{A_c^2}{4} \left(2 + m^2\right)\)

For detection, use only one sideband, amplitude \( mA_c / 2 \).

Useful power \( P_u = \frac{m^2 A_c^2}{8} \)

Efficiency at receiver \( \eta = \frac{P_u}{P_s} = \frac{m^2}{2(2 + m^2)} ; 0 < m^2 \leq 1 \)

Maximum efficiency \( \eta_{\text{max}} = \frac{1}{6} \cdot 100 \approx 16.67\% \) (m=1)

Power from both sidebands = useful , \( P_u = m^2 A^2 / 4 \),
double efficiency : \( \eta = \frac{m^2}{2 + m^2} , \eta_{\text{max}} = 33.33\% \)
observation

- This amplitude modulation is not linear

\[ s_1(t) = A_c \left[ 1 + k_a x_1(t) \right] \cos \omega_c t; \]
\[ s_2(t) = A_c \left[ 1 + k_a x_2(t) \right] \cos \omega_c t. \]

If \( s_{1+2}(t) \) results from the modulation with the sum \( x_1(t) + x_2(t) \) we have

\[ s_{1+2}(t) = A_c \left[ 1 + k_a \left[ x_1(t) + x_2(t) \right] \right] \cos \omega_c t \neq s_1(t) + s_2(t) \]

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**Linear Amplitude Modulation**

\[ s(t) = a(t) \cos \left[ \omega_c t + \phi(t) \right] \]
\[ = \left[ a(t) \cos \phi(t) \right] \cos \omega_c t - \left[ a(t) \sin \phi(t) \right] \sin \omega_c t \]
\[ = s_i(t) \cos \omega_c t - s_Q(t) \sin \omega_c t, \]

canonical form of a bandpass signal

\( s_i(t) \) – in phase component (I-channel),
\( s_Q(t) \) – in quadrature component (Q-channel).

linear modulation \( \iff s_i(t) \) and \( s_Q(t) \) - linear dependent on \( x(t) \)
### Linear Amplitude Modulation

<table>
<thead>
<tr>
<th>Modulation type</th>
<th>In phase component</th>
<th>In quadrature component</th>
<th>Observations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Double sideband suppressed carrier DSB-SC</td>
<td>$x(t)$</td>
<td>0</td>
<td>$x(t)$ - message</td>
</tr>
<tr>
<td>Single sideband SSB</td>
<td>$\frac{1}{2} x(t)$</td>
<td>$\frac{1}{2} \hat{x}(t)$</td>
<td>$\hat{x}(t) = \mathcal{H}{x(t)}$</td>
</tr>
<tr>
<td>Superior sideband is transmitted</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inferior sideband is transmitted</td>
<td>$\frac{1}{2} x(t)$</td>
<td>$-\frac{1}{2} \hat{x}(t)$</td>
<td>$\hat{x}(t) = \mathcal{H}{x(t)}$</td>
</tr>
<tr>
<td>with vestigial sideband VSB</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>the vestige of the superior sideband is transmitted</td>
<td>$\frac{1}{2} x(t)$</td>
<td>$\frac{1}{2} x'(t)$</td>
<td></td>
</tr>
<tr>
<td>the vestige of the inferior sideband is transmitted</td>
<td>$\frac{1}{2} x(t)$</td>
<td>$-\frac{1}{2} x'(t)$</td>
<td></td>
</tr>
</tbody>
</table>

DSB-SC – suppressed carrier, bandwidth is the same as full AM

VSB – large bandwidth signals

### Observations

- in phase component $s_i(t)$ depends only on the message signal
- in quadrature component $s_Q(t) =$ filtered version of the message signal. The spectral modification of $s(t)$ compared to $x(t)$ is given only by $s_Q(t)$
- The purpose of $s_Q(t)$ if it exists, is to interfere with the in phase component to eliminate / reduce the power from a sideband of the modulated signal
Double sideband-suppressed carrier modulation

\[ s(t) = A_c x(t) \cos(\omega_c t) \Rightarrow S(\omega) = \frac{A_c}{2} [X(\omega - \omega_c) + X(\omega + \omega_c)] \]

- carrier multiplied with the message signal
- carrier absent in the spectrum
- But \( S(\omega) \) has spectral components in \( \omega = \omega_c \)
- transmitted bandwidth = 2xbandwidth of the modulating wave
Coherent (Synchronous) Detection

- reconstruct modulating signal $x(t)$

\[ v(t) = s(t) \cos(\omega_c t + \theta) = A_x x(t) \cos \omega_c t \cos(\omega_c t + \theta) \]

\[ = \frac{A_x}{2} x(t) \cos \theta + \frac{A_x}{2} x(t) \cos(2\omega_c t + \theta) \]

base band $(-\omega_m, \omega_m)$ centered in $2\omega_c$, $(2\omega_c - \omega_m, 2\omega_c + \omega_m)$

$\Phi_{\omega_c}(t) = \frac{A_x}{2} x(t) \cos \theta$ (LPF output)

Desynchronisation between local oscillators - receiver & emission unit $\Rightarrow$ phase error $\theta \Rightarrow$ decreasing of detector response.

maximum for $\theta = 0$; zero for $\theta = \pm \frac{\pi}{2}$.

Local oscillator of the receiver synchronized with the local oscillator that generates the carrier signal in frequency and in phase.
Quadrature-Carrier multiplexing

• Also known as Quadrature-amplitude modulation (QAM)
• Transmit two DSB-SC modulated waves on the same bandwidth ⇒ bandwidth-conservation scheme
• Modulators with quadrature phase:
  – carriers in quadrature, same frequency, differ in phase by ±π/2 (±90°)
• Demodulator: two coherent detectors with 90 degree phase shift
• θ=±π/2: output of synchronous detector = null (quadrature effect)

\[ x_1(t), x_2(t) \text{ - independently modulating signals.} \]
\[ s(t) = A_c x_1(t) \cos \omega_c t + A_c x_2(t) \sin \omega_c t \]
Single side band modulation – SSB

• one sideband transmitted
• frequency-discrimination scheme with 2 steps
• Product modulator $\Rightarrow$ double sideband-suppressed carrier
• Bandpass filter: passes the sideband selected for transmission and suppresses the remaining sidebands
• separation lower and upper sideband $\Rightarrow$ energy gap in the spectrum of the message signal $x(t)$
• Speech signals (telephony): energy gap= -300, 300 Hz
• modulated signal: energy gap $2\omega_m$

Restrictions for BPF of sideband selection:
1. selected sideband $\subset$ passing band of the filter,
2. unwanted sideband $\subset$ stop band of the filter,
filter's transition bandwidth $< 2\omega_m$.
Demodulation- synchronous detection.
Vestigial sideband modulation VSB

Transmitted: modified version of one sideband and to compensate this, an appropriately chosen vestige of the other sideband.

Well suited for large bandwidth signals: commercial television

- The bandpass filter makes difference between SSB and VSB.

- Odd symmetry in the transition bandwidth $[f_c - f_v, f_c + f_v]$ centered on the cutoff frequency $f_c$

$$H(\omega - \omega_c) + H(\omega + \omega_c) = 1$$
• The sum of its magnitude at frequencies symmetrically with \( f_c \) is 1, in the transition bandwidth

\[
H(\omega - \omega_c) + H(\omega + \omega_c) = 1
\]

• Phase is linear

• The transmission bandwidth is

\[
B_T = B + 2\pi f_v
\]

\( B \)-message bandwidth; \( f_v = \frac{\omega_c}{2\pi} \)-vestigial bandwidth

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\[
s(t) = \frac{1}{2} A_c x(t) \cos \omega_c t \pm \frac{1}{2} A_c x'(t) \sin \omega_c t
\]

• “+” is for transmitting a vestige from the upper sideband

• “-” is for the lower sideband vestige

\( x'(t) \) – in quadrature component of the signal \( s(t) \) obtained by filtering \( x(t) \) with \( H_Q(\omega) \)

\[
H_Q(\omega) = j[H(\omega - \omega_c) + H(\omega + \omega_c)]; \quad -B \leq \omega \leq B
\]
• SSB modulation can be seen as VSB with vestige reduced to zero
• The filter for the in quadrature component:
  \[ H_Q(\omega) = -j \text{sgn } \omega \]
• Or
  \[ x'(t) = \mathcal{H}\{x(t)\} \]
• The video signal bandwidth is large with significant low frequencies spectral components. Hence the VSB
• Demodulation circuits must be simple (affordable). This requirement imposes envelope detection hence transmitting the carrier besides the VSB signal
• In reality, since at transmitter the power is high, the VSB filter is used at receiver (low power, relatively affordable filter)
• North America: channel bandwidth 6 MHz
• Picture carrier: 55.25 MHz
• Sound carrier: 59.75 MHz
• Image signal spectrum is 1.25 MHz below carrier, and 4.5 MHz above it

Adding the carrier:

\[ s(t) = A_i \left[ 1 + \frac{1}{2} mx(t) \right] \cos \omega_c t \pm \frac{1}{2} mA_s x'(t) \sin \omega_c t \]

- \(m\)-modulation degree.
- At envelope detection:

\[ a(t) = A_i \left[ 1 + \frac{1}{2} mx(t) \right] \sqrt{1 + \left[ \frac{1}{2} \frac{mx'(t)}{1 + \frac{1}{2} x(t)} \right]^2} \]
• The signal is distorted at the receiver. This can be reduced by
  - reducing the modulation degree
  - increasing vestigial bandwidth to reduce \( x'(t) \)

The vestigial bandwidth 0.75MHz (1/6 of the bandwidth) is chosen such that distortion is acceptable even for \( m=100\% \)

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**Frequency translation**

- Change the carrier frequency of the modulated signal from \( \omega_1 \) to \( \omega_2 \)
- Mixer: product modulator+bandpass filter

Up conversion \( \omega_2 > \omega_1 \)
\[ \omega_2 = \omega_1 - \omega_0 \]

Down conversion \( \omega_2 < \omega_1 \)
\[ \omega_2 = \omega_1 - \omega_0 \]
Up conversion, $\omega_2 > \omega_1$

Image signal spectrum
Spectrum of the modulated signal with up conversion

Down conversion, $\omega_2 < \omega_1$

Spectrum of the modulated signal with down conversion
Image signal spectrum
Frequency Division Multiplexing

- Telephony systems: 300Hz-3400Hz
- Goal: transmit simultaneously several vocal signals on the same channel:
  - FDM-frequency division multiplexing
  - TDM-time division multiplexing
- FDM, using AM-SSB
- Distance between carriers 4kHz
- BPFs – bandwidth limitation at 4kHz

LPF - remove high frequency components
AM Modulators modulate the signals on different carrier frequencies
Angular Modulation

- modulate a carrier: alter its angle—phase, according to the message; amplitude ~ constant

- **Advantage**: signal more robust against noise and interference.

- **Disadvantage**: increase in bandwidth
Angular Modulation

Modulated signal - rotating vector with amplitude $A_c$ and angle $\theta_i(t)$:

$$s(t) = A_c \cos \theta_i(t)$$

Its angular velocity: **instantaneous frequency of the modulated signal.**

$$\omega_i(t) = \frac{d\theta_i(t)}{dt}$$

Phase modulation (PM)

$$\theta_i(t) = \omega_i t + k_p x(t)$$

$k_p$ [rad/V] - phase sensitivity.

Frequency modulation (FM)

$$\omega_i(t) = \omega_c + 2\pi k_f x(t)$$

$k_f$ [Hz/V] - frequency sensitivity.

$$\theta_i(t) = \omega_i t + 2\pi k_f \int_0^t x(\tau) d\tau$$

$$\Rightarrow s(t) = A_c \cos \left[ \omega_i t + 2\pi k_f \int_0^t x(\tau) d\tau \right]$$

FM signal generated using $x(t)$ = PM signal generated using $\int_0^t x(\tau) d\tau$. 

FM signal generated using $x(t)$ = PM signal generated using $\int_0^t x(\tau) d\tau$. 

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Frequency Modulation

modulating signal: \( x(t) = A_m \cos \omega_m t \)

instantaneous frequency \( \omega_i (t) = \omega_c + 2\pi k / A_m \cos \omega_m t \)

The frequency deviation \( \Delta \omega = 2\pi k / A_m \)

is the maximum instantaneous difference between FM modulated carrier frequency and nominal carrier frequency;

The modulation index \( \beta = \frac{\Delta \omega}{\omega_m} = \frac{2\pi k / A_m}{\omega_m} \)

\[ s(t) = A_c \cos \theta(t) = A_c \cos [\omega_0 t + \beta \sin \omega_m t] \]

Essential characteristic for FM: frequency deviation \( \Delta f \) is proportional with modulating signal amplitude \( A_m \); does not depend on its frequency.

\( \beta << 1 \) radian - **narrow band modulation.**

\( \beta >> 1 \) radian - **wide band modulation.**

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Narrow Band Frequency Modulation

\[ s(t) = A_c \cos \omega_c t \cos (\beta \sin \omega_m t) - A_c \sin \omega_c t \sin (\beta \sin \omega_m t) \]

If \( \beta < \frac{\pi}{36} \) rad \( \Rightarrow \cos (\beta \sin \omega_m t) \approx 1 \) and \( \sin (\beta \sin \omega_m t) \approx \beta \sin \omega_m t \)

\( \Rightarrow s(t) = A_c \cos \omega_c t - \beta A_c \sin \omega_c t \sin \omega_m t \).
Phasorial representation of the FM and AM signals

Narrow band FM and AM – same bandwidth

Narrow Band FM Spectrum – general case

\[ S(\omega) = \pi A_c \left[ \delta(\omega - \omega_c) + \delta(\omega + \omega_c) \right] + \pi A_t \left[ \frac{X(\omega - \omega_c)}{\omega - \omega_c} - \frac{X(\omega + \omega_c)}{\omega + \omega_c} \right] \]

\[ s(t) = A_c \cos(\omega_c t) \int_0^{y(t)} x(\tau) d\tau \]

\[ = A_c \cos(\omega_c t) \cos(2\pi k_f y(t)) - A_c \sin(\omega_c t) \sin(2\pi k_f y(t)) \]

Narrow band modulation, \(2\pi k_f A \leq \frac{\pi}{36} \Rightarrow \)

\[ s(t) \cong A_c \cos(\omega_c t) - A_c 2\pi k_f y(t) \sin(\omega_c t) \]

Ex: \(\pi/10 = 0.314\), \(\sin(\pi/10) = 0.309\)
Wide Band Frequency Modulation

\[ s(t) = A_c \cos \left( \omega_c t + \beta \sin \omega_m t \right) \]

\[ = A_c \cos \omega_c t \cos \left( \beta \sin \omega_m t \right) - A_c \sin \omega_c t \sin \left( \beta \sin \omega_m t \right) \]

\[ \Rightarrow s(t) = \text{Re} \left\{ A_c e^{i(\omega_c t + \beta \sin \omega_m t)} \right\} = \text{Re} \left\{ \tilde{s}(t) e^{j\omega_c t} \right\}, \]

\[ \tilde{s}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{jn\omega_m t} \] - complex envelope of the FM signal \( s(t) \)

\( J_n(x) \) - Bessel function of first kind, order \( n \) and variable \( x \).

\[ s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c t + n\omega_m t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos 2\pi (f_c + n f_m) t \]

\[ S(\omega) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta) \left[ \delta(\omega - \omega_c - n\omega_m) + \delta(\omega + \omega_c + n\omega_m) \right]. \]
properties of Bessel's functions
1. $J_n(\beta) = (-1)^n J_{-n}(\beta)$ for any $n \in \mathbb{Z}$,
2. For small $\beta$, we have:
   
   \[
   J_0(\beta) \equiv 1 ; \quad J_1(\beta) \equiv \frac{\beta}{2} ; \quad J_2(\beta) \equiv 0 ; \quad n > 2 ; \quad |\beta| \ll 1 ;
   \]
3. $\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$.

First five Bessel functions, $J_0(\beta)$-$J_4(\beta)$

\[
S(\omega) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta) \left[ \delta (\omega - \omega_c - n\omega_m) + \delta (\omega + \omega_c + n\omega_m) \right].
\]
Remarks
1. FM Spectrum: component on the carrier, $\omega_c$ and an infinite set of components on the sidebands at a distance of $\omega_m$, $2\omega_m$, ..., $\pm \omega_c$
2. $|\beta| \ll 1$ (narrow bandwidth FM), only $J_0(\beta)$ and $J_1(\beta)$ have significative values $\Rightarrow$ carrier ($\omega_c$) and two lateral bands $\omega_c \pm \omega_m$.
3. The amplitude of the component on $\omega_c$ depends on the factor $J_0(\beta)$ $\Rightarrow$ not constant.

The power is constant:

\[
P = \frac{1}{2} A_c^2 \sum_{n=-\infty}^{\infty} J_n^2(\beta) = \frac{1}{2} A_c^2
\]

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Example 1

The amplitude of the modulating signal affects the FM spectrum.

\[ \beta = \frac{\Delta \omega}{\omega_m} = \frac{2\pi k_f A_m}{\omega_m} \]

- \( f_m = \) const;
- \( A_m \) variable \( \Rightarrow \Delta f = k_f \cdot A_m \) variable
- \( \Rightarrow \beta \) variable
- Spectral components separated by \( f_m \) (const).

Example 2

The frequency of the modulating signal affects the FM spectrum.

\[ \beta = \frac{\Delta \omega}{\omega_m} = \frac{2\pi k_f A_m}{\omega_m} \]

- \( A_m = \) const \( \Rightarrow \Delta f = k_f \cdot A_m \) const
- \( f_m \) variable \( \Rightarrow \beta = \) variable
- + number of spectral components in the interval \([f_c - \Delta f, f_c + \Delta f]\)
- increases
- FM bandwidth \( \overrightarrow{\beta \rightarrow \infty} 2\Delta f \)
The transmission bandwidth of FM signals

\[ S(\omega) = \frac{A}{2} \sum_{n=-\infty}^{\infty} J_n(\beta) \left[ \delta(\omega - \omega_c - n\omega_m) + \delta(\omega + \omega_c + n\omega_m) \right] . \]

For \( \beta \to \infty \), the transmission bandwidth \( B_r \to 2\Delta f_c \); centered on \( f_c \).

**Carson's rule**: nearly all (~98%) of the power of a FM signal lies within a bandwidth \( B_r \) of:

\[ B_r \cong 2\Delta f + 2f_m = 2\Delta f \left(1 + \frac{1}{\beta} \right) \]

**Carson's rule**: under – estimation of transmission band.

The **universal curve**: over – estimation of transmission band.

The transmission bandwidth is found between the two estimates

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**Equivalent definition of the transmission bandwidth**

The frequency interval where the spectral components of the FM signal have a value superior to 1% of the carrier amplitude.

\[ B_r = 2n_{\text{max}}f_m \]

where for each \( n \leq n_{\text{max}} \) is satisfied the condition

\[ |J_n(\beta)| > 0.01. \]

The value \( n_{\text{max}} \) depends on \( \beta \).

\[ \begin{array}{c|c|c|c|c|}
\beta & 2n_{\text{max}} & \beta & 2n_{\text{max}} \\
0.1 & 2 & 0.5 & 16 \\
0.3 & 4 & 0.5 & 28 \\
0.5 & 4 & 1.0 & 50 \\
1 & 6 & 2.0 & 70 \\
2 & 8 & \end{array} \]
Non harmonic modulating wave

\( x(t) \) - modulating signal, maximum frequency \( W \) (same as \( f_m \))

\[ A_{\text{max}} = \max |x(t)| \Rightarrow \Delta f = k_f A_{\text{max}}, \text{ frequency deviation} \]

\[ \Rightarrow D = \Delta f / W \] \( \text{deviation ratio} \) (same as \( \beta \)).

Carson's rule: replace \( \beta \) with \( D \) and \( f_m \) with \( W \) and the universal curve for any modulating signal

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Example 3

North America, radio transmissions:

\[ \Delta f = 75 \text{ kHz} ; \ W = 15 \text{ kHz} ; \ D = \frac{75}{15} = 5. \]

Carson's rule: \( B_f = 2(\Delta f + W) = 180 \text{ kHz} \).

Universal curve: \( D = 5 \Rightarrow B_f = 3,2\Delta f = 240 \text{ kHz} \).

In practice a transmission bandwidth of 200 kHz is used.
Frequency Modulated Signals’ Generation

There are 2 methods,

direct - based on a voltage controlled oscillator - 555 timer
indirect - 1. narrow band FM
    2. frequency multiplication to set the frequency deviation.

The second method ⇒ high frequency stability
⇒ FM radio broadcasting

FM Signal Generation, indirect method

The frequency deviation is small to reduce distortions in narrow band modulation

narrow band FM signal ⇒ wide band FM signal by frequency multiplication
Input: \( s(t) = A \cos \left( \omega t + 2\pi k t \int_0^t x(\tau) d\tau \right) \), with \( f(t) = f_e + k_j x(t) \)

Output: \( v(t) = a_1 s(t) + a_2 s^2(t) + \ldots + a_n s^n(t) \)

The pass-band of the band-pass filter is \( n \) times larger than of the bandwidth of the signal \( s(t) \).

\[ s'(t) = A \cos \left( n\omega t + 2\pink t \int_0^t x(\tau) d\tau \right) \]

The instantaneous frequency: \( f'(t) = nf_e + nk_j x(t) \).

- The **frequency multiplier** is a nonlinear device followed by a bandpass filter
- The nonlinear device is **memoryless** in the sense that it doesn’t have in its structure reactive elements
Demodulation

- Reconstruction of modulating wave
- Inverse characteristic of transfer of the characteristic of transfer of the FM modulator
- 1. directly: frequency discriminator: output proportional with the instantaneous frequency of the FM signal.
- 2. indirectly: PLL circuit (Phase-locked loop)

FM quadrature demodulator

The block diagram of the demodulator
• The quadrature demodulator converts the FM signal:

\[
s(t) = A \cos \left(2\pi f_c t + 2\pi \int_0^t x(\tau) d\tau \right)
\]

\[f_c = 10.7\text{MHz} = 10700\text{kHz}\]

into a PM signal, and a PM detector is used to recover the message signal, \(x(t)\)

• 1. The phase shifter converts FM modulation into PM modulation but preserves the FM modulation

• 2. The analog multiplier serves as a phase detector, PD, and produces an output being linearly proportional to PM. PD is not sensitive to FM

• 3. The low-pass filter suppresses the spectral components with high frequency \((2f_c)\)
The phase shift is linearly proportional to the instantaneous frequency deviation around the carrier frequency, 10700kHz.

\[ \phi^\prime(f) = -90^\circ + \frac{34.42}{150} (f - 10700) \]

\[ \phi(f) = -\frac{\pi}{2} + 4 \cdot 10^{-3} (f - 10700) \text{ [rad], } f \text{ [kHz]} \]

The phase shifted signal is:

\[ \tilde{s}(t) = \tilde{A} \cos \left[ 2\pi 10700t + 2\pi k \int_0^t x(\tau) d\tau + 4 \cdot 10^{-3} (f - 10700) - \frac{\pi}{2} \right] \]

\[ = \tilde{A} \sin \left[ 2\pi 10700t + 2\pi k \int_0^t x(\tau) d\tau + 4 \cdot 10^{-3} (f - 10700) \right] \]
But the amplitude response of the phase shifter is

\[ \text{FM} \]

\[ 20 \log \frac{A_m}{A} = 20 \log \frac{\tilde{A}_M}{\tilde{A}_m} = 0.771 \text{ [dB]} \]

Maximum gain

The amplitude varies only a little, and therefore we can consider the output amplitude of the output from the phase shifter is constant:

\[ \frac{\tilde{A}_M}{\tilde{A}_m} \simeq 1.093 \Rightarrow \tilde{A} \simeq \text{cst}. \]

The mean gain can be taken as:

\[ 20 \log \frac{\tilde{A}}{A} \simeq -26.57 \text{ [dB]} \Rightarrow \tilde{A} \simeq 4.7 \cdot 10^{-2} A \]
The phase detector is implemented by an analog multiplier:

\[ s(t) = A(A_t \cos \left( 2\pi 10700t + 2\pi \int_0^t x(\tau) d\tau \right) \sin \left( 2\pi 10700t + 2\pi k \int_0^t x(\tau) d\tau + 4 \cdot 10^{-3} (f - 10700) \right) \]

\[ = 2.35 \cdot 10^{-2} A^2 \sin \left( 4 \cdot 10^{-3} (f - 10700) \right) \]

The low-pass filter suppresses the second component, centered at 21.4 MHz. The first component, a base-band component is retained:

\[ \tilde{x}(t) = 2.35 \cdot 10^{-2} A^2 \sin \left( 4 \cdot 10^{-3} (f - 10700) \right) \]

\[ \left| 4 \cdot 10^{-3} (f - 10,700) \right| \leq 4 \cdot 10^{-3} \cdot 75 = 0.3 \text{ [rad]} \]

For \( |\alpha| \leq 0.3 \) \( \sin \alpha \cong \alpha \)

\[ \tilde{x}(t) = 2.35 \cdot 10^{-2} A^2 4 \cdot 10^{-3} (f - 10700) \]

\[ = 94.12 \cdot 10^{-6} A^2 (f - 10700) \]
• The instantaneous frequency is
  \[ f = 10700 + kx(t) \text{ [kHz]} \]
• And therefore
  \[ \hat{x}(t) \approx 94.2 \times 10^{-6} A^2 kx(t) \]
• We have obtained a FM demodulator. The circuit configuration presented is almost exclusively used to implement a modern FM demodulator (discriminator).

• The transfer function obtained is called an S curve

\[ v_{out} \]
\[ f_i \]

Linear portion of the characteristic

10.7 [MHz]

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Stereo FM Signals Multiplexing

Stereo - 2 different signals are transmitted using the same carrier. The stereo radio broadcasting satisfies the conditions:
1. It is realized inside the broadcasting FM channel allocated,
2. It is compatible with the mono receivers.

The signal \(x_r(t) + x_l(t)\) represents the part of the base-band disponible for mono reception.
The signal \(x_r(t) - x_l(t)\) is amplitude modulated with 2 sidebands and suppressed carrier. The multiplexed signal:
\[x(t) = \left[x_r(t) + x_l(t)\right] + \left[x_r(t) - x_l(t)\right] \cos 4\pi f_p t + K \cos 2\pi f_p t,\]
is frequency modulated.
Non-linear Effects in Frequency Modulation

- Nonlinearities in electronic circuits
  - Strong non-linearity which is intentional, for given applications
- Weak non-linearity
- Effect of weak non-linearity on FM systems
Non-linear Effects in Frequency Modulation

Consider a non-linear communication channel with the input-output transfer characteristic:

\[ v_0(t) = a_1v_1(t) + a_2v_2(t) + a_3v_3(t) , \]

having at its input the frequency modulated signal:

\[ v_i(t) = A_c \cos \left( 2\pi f_c t + \phi(t) \right) ; \quad \phi(t) = 2\pi k_j \int_0^t x(\tau) d\tau \]

\[ \Rightarrow v_0(t) = a_1A_c \cos \left( 2\pi f_c t + \phi(t) \right) + a_2A_c^2 \cos^2 \left( 2\pi f_c t + \phi(t) \right) + \]

\[ + a_3A_c^3 \cos^3 \left( 2\pi f_c t + \phi(t) \right) . \]

From the trigonometric relations:

\[ \cos^2 x = \frac{1 + \cos 2x}{2} ; \quad \cos^3 x = \frac{3x + 3\cos x}{4} \]

we have:

\[ v_0(t) = \frac{a_2A_c^2}{2} + \left( a_1A_c + \frac{3}{4}a_3A_c^3 \right) \cos \left[ 2\pi f_c t + \phi(t) \right] + \]

\[ + \frac{a_3A_c^4}{4} \cos \left[ 6\pi f_c t + 3\phi(t) \right] . \]

For the detection of the FM signal from \( v_0(t) \) it is necessary its identification.
Let $\Delta f$ be the frequency deviation of the FM signal and $W$ the maximum frequency of the modulator signal. Applying Carson's rule we have the separation condition:

$$2f_c - (2\Delta f + W) > f_c + (\Delta f + W) \Rightarrow f_c > 3\Delta f + 2W.$$  

If this condition is satisfied then we can extract from $v_0(t)$, using a band-pass filter with central frequency $f_c$ and bandwidth $2\Delta f + 2W$, the term

$$v_0'(t) = \left(a_4A_c + \frac{3}{4}a_3A_c^3\right)\cos\left[2\pi f_c t + \phi(t)\right].$$

The Super-heterodyne Receiver

A radio broadcasting receiver has not only the goal to demodulate the received signal. Other goals:

- Selection of the desired carrier frequency,
- Filtering, for the separation of the desired signal from other modulated signals,
- Amplification, for the compensation of the losses produced by the propagation.
\( f_{RF} = 630 \text{ kHz} \)

Radio Timisoara

A IF signal is generated in the receiver if the difference of the local oscillator frequency and of the input carrier frequency equals \( \pm f_{IF} \):

\[
f_{RF} = f_{LO} \pm f_{IF}.
\]

only one of these frequencies corresponds to the carrier, the other one is named **image frequency** \( f_{RF} + 2f_{IF} \).
• For FM case, after the IF amplifier there is limiter and a bandpass filter
• Detection is made using a frequency discriminator