Relativistic Doppler Effect Free of "Plane Wave" and "Very High" Frequency Assumptions

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We show that a free of assumptions approach to the Doppler effect (plane wave and "very small" period assumptions) leads to a Doppler factor which depends on the involved frequencies. The result is that the Doppler effect shifts differently the different frequencies present in the studied electromagnetic radiation.

1. Introduction

The Doppler factor formula is one of the most frequently derived equations in classical and in relativistic physics as well. In many cases, we make assumptions concerning the magnitude of the involved frequencies, the source-receiver distance and the character of the wave (plane or spherical).

We consider that in a Doppler Effect experiment we compare the time interval between the emissions of two successive wave-crests by the source (emission period), with the time interval between theirs reception, by the observer (reception period) [1]. The emission period is measured in the rest frame of the source (proper period of emission). The reception period is measured in the observer's rest frame (proper period of reception). A Doppler factor formula relates the two periods mentioned above via a Doppler factor. The Doppler factor depends on the followed scenario, which prescribes the relative motion of receiver and source, and on the assumptions made concerning the magnitude of the physical quantities related by it.

We consider that speaking about a physical quantity it is advisable to define the observer who measures it, when and where he performs the measurement and the measuring devices involved in the measurement. We measure time intervals as a difference between the readings of the same clock (proper time interval) or as a difference between the readings of two distant synchronized clocks, of the same inertial reference frame. In a Doppler Effect experiment performed in an electromagnetic wave, we consider the rest frame of the source ($K_S\left(X_S O_S Y_S\right)$) and the observer's rest frame ($K_R\left(X_R O_R Y_R\right)$) as well. The axes $O_S X_S$ and $O_R X_R$ are overlapped, the corresponding axes are parallel to each other and the relative motion takes place in the positive direction of the overlapped axes. At each point of the plane defined by the axes of the mentioned reference frames, we find instantaneously two clocks belonging to the involved reference frames, each synchronized in its rest frame in accordance with a synchronization procedure proposed by Einstein [2]. The involved clocks are $C_S\left(0,0\right)$ and $C_R\left(0,0\right)$ located at the origins O_S, and O_R.

They read $t_S = t_R = 0$ when the origins mentioned above are located at the same point in space. An important consequence of the scenario is that at a given time we find at a given point in space a clock $\mathrm{C_S}(x_S, y_S)$ of $\mathrm{K_S}(\mathrm{X_S O_S Y_S})$ reading t_S and a clock $\mathrm{C_R}(x_R, y_R)$ of $\mathrm{K_R}(\mathrm{X_R O_R Y_R})$ reading t_R.

We consider separately two cases: The case when source S is at rest and observer R moves relative to it and the case when R is at rest and S moves relative to it. In the case of a scenario which involves a stationary source S and a uniformly moving receiver R, the Doppler formula

$$D = \frac{T_S}{T_R} = \frac{f_R}{f_S} = \frac{1 - \frac{\vec{\mathrm{v}}_R \cdot \vec{c}}{c^2}}{\sqrt{1 - \left(\frac{\mathrm{v}_R}{c}\right)^2}} \tag{1}$$

is in large use. T_S represents a proper time interval measured as a difference between the readings of clock $\mathrm{C_S}(0,0)$ attached to the stationary source. T_R represents a proper time interval as well measured as a difference between the readings of a clock $\mathrm{C_R}$ attached to observer R. $\vec{\mathrm{v}}_R$ represents the instantaneous velocity of R and D represents a Doppler factor. Since it is independent of involved periods (frequencies $f_S = 1/T_S$ and $f_R = 1/T_R$) the Doppler factor defined by Eq.(1) is linear. Consequently, if S emits a radiation in which more frequencies are present, the supposed linearity of Doppler shift makes that all the present frequencies are shifted by the same Doppler factor, characteristic for the followed scenario. It is surprising that the Doppler factor (1) does not depend on the source-observer distance.

Authors derive Eq.(1) taking as a starting point the relativistic invariance of the phase of a plane electromagnetic wave [3, 4]. The concept of plane wave is a mathematical construction with not much physical support. It is associated with the "very large" source-receiver distance assumption. Using Eq.(1) many authors fail to mention that fact and use it to support special relativity [5].

The purpose of our paper is to derive Doppler formulas free of assumptions concerning how small or how large are the involved physical quantities. Such derivations show that Eq.(1) holds only in the case of "very small" period ("very high" frequency) assumption. The "very small" period assumption is associated with the concept of locality in the period measurement by a moving receiver. Making the locality assumption, we consider that the reception period is small enough in order to consider that R receives two successive wave crests from the same point in space. Moreau [6] introduced the concept of non-locality in connection with an accelerating receiver who performs the hyperbolic motion. Measuring the reception period as a time interval between the receptions of two successive wave crests, we infer that during that time interval R does not have enough information in order to reckon the reception period.

2. A realistic approach to the Doppler effect

2.1. Stationary source and moving observers

The scenario we propose involves a source S of electromagnetic waves, located at the origin O_S of its rest frame $K_S\left(X_S O_S Y_S\right)$. An observer R moves with constant velocity $v_R = v$ parallel to the $O_S X_S$

axis (to the ground), at an altitude d. We consider the problem in the rest frame of the source. In order to find out the result of the measurements performed by R we use the time dilation formula. $\mathrm{R}_0(0,d)$ represents an instantaneous position of R when the clocks of the $\mathrm{K_S}(\mathrm{X_S O_S Y_S})$ frame read $t_S = 0$. He receives there a wave crest emitted by S at $-d/c$. $R_n(\vec{r}_n)$ represents a later instantaneous position of R where he receives the *n*-th emitted wave crest at a time t_n, measured by S, at a distance $\mathrm{v}t_n$ from its initial position (Figure 1).

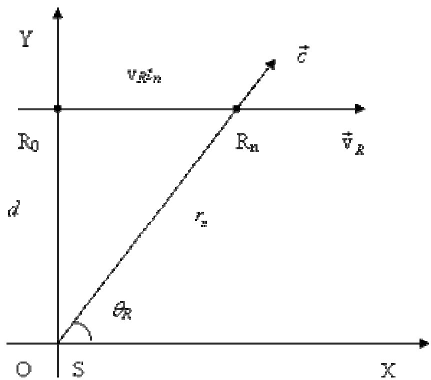

Figure 1: Doppler effect experiment. It involves a stationary source of light located at the origin of its rest frame. An observer R moves with constant velocity v parallel to the OX axis (to the ground) at an altitude *d*.

Pythagoras' theorem applied to Figure 1 leads to

$$d^2 + \mathrm{v}^2 t_n^{\ 2} = c^2\left(t_n + \frac{d}{c} - nT_S\right)^2 . \tag{2}$$

Solved for t_n Eq.(2) leads to

$$t_n = T_S \frac{\left(n - \frac{d}{c} f_S\right) + A_n}{1 - \left(\frac{\mathrm{v}}{c}\right)^2}, \tag{3}$$

where

$$A_n = \sqrt{\frac{d^2}{c^2} f_S{}^2 + n\frac{\mathrm{v}^2}{c^2}\left(n - \frac{2d}{c} f_S\right)}. \tag{4}$$

A clock $\mathrm{C}\left(\vec{r}_n\right)$ attached to R and reading $t_R = 0$ at location R_0, reads $t_{R,n}$ at location R. We have in accordance with the time dilation effect

$$t_n = \frac{t_{R,n}}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}} \tag{5}$$

and the proper time in $\mathrm{K_R}$ when R receives the *n*-th crest is

$$t_{R,n} = T_S \frac{n - \frac{df_S}{c} + A_n}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}. \tag{6}$$

The proper reception period $T_R = t_{R,n} - t_{R,n-1}$, measured in the observer's rest frame, and the proper emission period T_S, measured in the rest frame of the source, are related by

$$D = \frac{T_S}{T_R} = \frac{f_R}{f_S} = \frac{\sqrt{1-\left(\frac{\mathrm{v}}{c}\right)^2}}{1 + A_n - A_{n-1}}. \tag{7}$$

A first important conclusion is that the Doppler factor D is non-linear, as depending on f_S. The result is that if the radiation emitted by S is not monochromatic, the Doppler Effect shifts differently the components of different frequencies. In order to illustrate that fact consider that S represents a radio station which emits frequencies comprised between $16\ \mathrm{Hz} < f_S < 20000\ \mathrm{Hz}$ or even higher, R being located on a jet plane which can move at supersonic velocities. We have in mind the fact that the acoustic signals are transformed into electromagnetic ones. The electromagnetic wave carries them at a velocity c. The Doppler Effect shifts those frequencies in the same way as the electromagnetic ones. We present in Figure 2 the variation of D with n for constant values of d and v.

For $f_S \to \infty$ (very high frequency assumption) Eq.(7) becomes

$$D_{f_S \to \infty} = \frac{1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}. \tag{8}$$

For $d \to \infty$ (plane wave assumption) Eq.(7) becomes

$$D_{d \to \infty} = \frac{1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}. \tag{9}$$

For $n \to \pm\infty$ (longitudinal Doppler Effect) Eq.(7) becomes

$$D_{n\to\pm\infty} = \sqrt{\frac{1\mp \mathrm{v}c^{-1}}{1\pm \mathrm{v}c^{-1}}}. \tag{10}$$

As we see, the plane wave assumption and the very high frequency assumption lead to the same result. Because the first wave crest was emitted in the positive direction of the OY axis ($\theta_R = 90^0$), we can consider that in the plane wave assumption R moves in a plane wave in which the rays are parallel to the OY axis with Eqs.(8) and (9) describing the transversal Doppler shift.

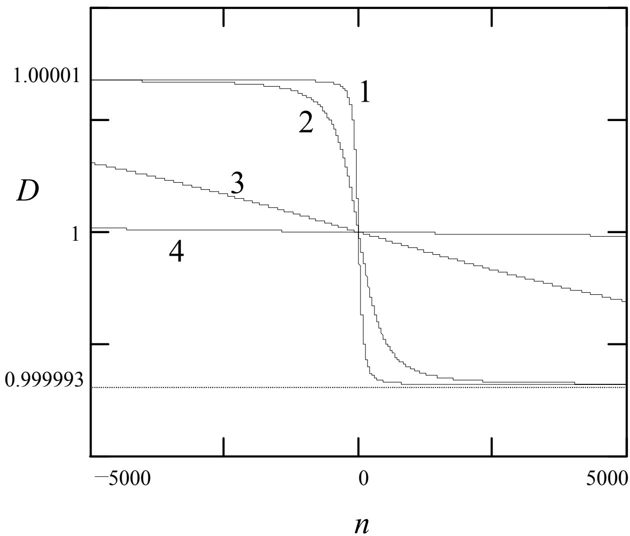

Figure 2.a

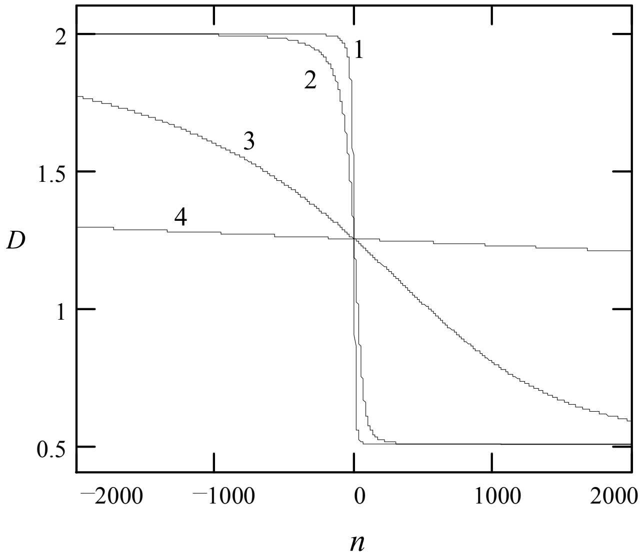

Figure 2.b

Figure 2: The variation of the Doppler factor *D* defined by Eq.(7) following the scenario presented in Figure 1, with the order number *n* of the emitted wave crest and for different values of the frequency f_S emitted by the source: 1-20 Hz; 2-100 Hz; 3-2000 Hz; 4-40000 Hz. **a)** R moves with supersonic velocity v=6x340 m/s at an altitude $d = 10^4\,\mathrm{m}$; **b)** R moves with relativistic velocity $\mathrm{v} = 0.6c$ at an altitude $d = 1.5\times 10^8\,\mathrm{m}$.

2.2. Stationary observers and moving source

We consider the experiment in the observer's rest frame. The source S moves with constant velocity $\mathrm{v}_S = \mathrm{v}$ parallel to the $\mathrm{O_R X_R}$ axis (to the ground), at an altitude d. In a free of assumptions approach (Figure 3), $\mathrm{S_0}$ represents an instantaneous position of S at $t_S = 0$, when it emits a wave crest received by R at d/c. S_n represents a later position of S at time nT'_S, measured in K_R, when it emits the *n*-th wave crest at a distance $nT'_S v$ at from S_0.

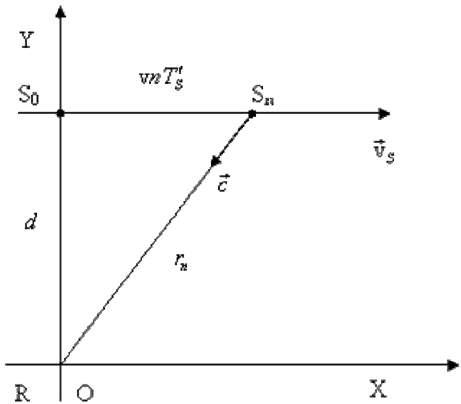

Figure 3: Doppler effect experiment. It involves a stationary observer located at the origin of its rest frame. A source S moves with constant velocity parallel to the OX axis (to the ground) at an altitude *d*.

The *n*-th wave crest is received by R at t_n given by

$$t_n = nT'_S + \frac{r_n}{c} = nT'_S + \sqrt{\frac{d^2}{c^2} + \frac{n^2 T'^2_S v^2}{c^2}}. \tag{11}$$

A clock moving with S measures between the emissions of two successive wave crests a proper time interval T_S related to T'_S by

$$T'_S = \frac{T_S}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{12}$$

so one can write

$$t_n = T_S \frac{n + B_n}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{13}$$

where

$$B_n = c^{-1}\sqrt{(nv)^2 + (df_S)^2(1 - v^2c^{-2})} \tag{14}$$

The time interval between the receptions of two successive wave crests is

$$T_R = t_n - t_{n-1} = T_S \frac{1 + B_n - B_{n-1}}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{15}$$

We are now able to define a Doppler factor characteristic for the followed scenario

$$D = \frac{T_S}{T_R} = \frac{f_R}{f_S} = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 + B_n - B_{n-1}} \tag{16}$$

As expected, we have in the case of the very high frequency and plane wave assumptions

$$D_{d\to\infty, f_S\to\infty} = \sqrt{1 - \frac{v^2}{c^2}} \tag{16}$$

A clock moving with S measures between the emissions of two successive wave crests a proper time interval T_S related to T'_S by

$$D = \frac{T_S}{T_R} = \frac{f_R}{f_S} = \frac{\sqrt{1-\frac{v^2}{c^2}}}{1+B_n - B_{n-1}} \tag{17}$$

whereas in the case of the longitudinal Doppler Effect we have

$$D_{n\to\pm\infty} = \sqrt{\frac{1\pm vc^{-1}}{1\mp vc^{-1}}} \tag{18}$$

In order to illustrate the results obtained above we present in Figure 4.a. the variation of *D* with *n* in the case when S moves with supersonic velocities at low altitudes, whereas in Figure 4.b we consider the case when S moves with relativistic velocities but at astronomic altitudes, a situation of interest for astronomers.

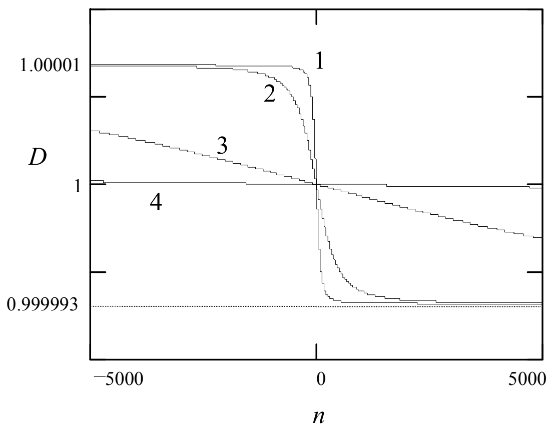

Figure 4.a

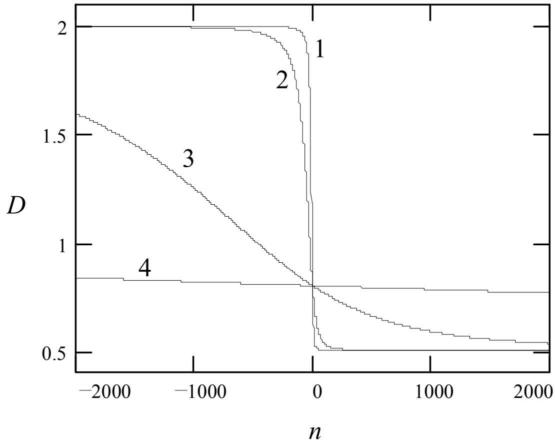

Figure 4.b

Figure 4: The variation of the Doppler factor D defined by Eq.(16) with the order number of the emitted wave crest for different values of the emitted frequency fS : 1 - 20 Hz; 2 - 100 Hz; 3 - 2000 Hz; 4 - 40000 Hz; **a)** Source S moves with supersonic velocity v=6x340 m/s at an altitude d=104 m; **b)** Source S moves with relativistic velocity v=0.6xc at an altitude d=1.5x108m.

3. A jump into the future

We consider now that there are objects that can fly with relativistic velocities at low altitudes. Such scenarios lead to high Doppler shifts and illustrate in a convincing manner the way in which the Doppler Effect shifts differently the different frequencies present in the electromagnetic radiation. We present in Figure 5.a the variation of D with the order number n of the emitted wave crest in the case of a stationary source and a uniformly moving observer. His motion takes place at an altitude $d = 10^4$ m, at a velocity $0.6c$.

In order to illustrate the fact that during the reception of two successive wave crests R does not have enough information in order to reckon the period we mark the points which lead to a calculated value of *D*. We present in Figure 5.b the variation of *D* with *n* in the case of the scenario which involves a stationary observer and a source moving with velocity $v = 0.6c$ at an altitude $d = 10^4$ m. The calculated values of *D* are marked in order to illustrate the fact that, during the reception of two successive wave-crests, R does not have enough information in order to reckon the Doppler factor *D*.

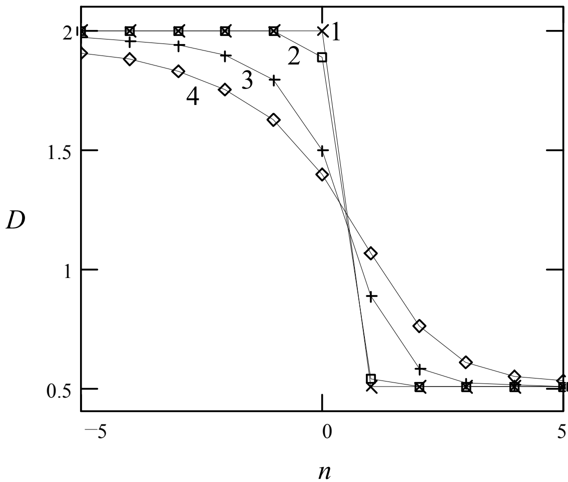

Figure 5.a

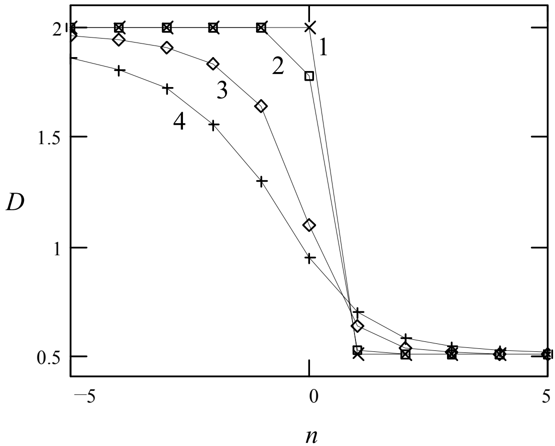

Figure 5.b

Figure 5: The variation of the Doppler factor *D* with the order number *n* of the emitted wave crest for different values of the emitted frequency f_S : 1 - 20 Hz; 2 - 2000 Hz; 3 - 20000 Hz; 4 - 40000 Hz. In order to illustrate the fact that between the reception of two successive wave crests the observer does not have enough information to reckon the frequency we have marked the time when a value of the frequency is calculated. **a)** the case of a stationary source and an observer moving with relativistic velocity $\mathrm{v} = 0.6c$ at low altitude $d = 10^4\,\mathrm{m}$; **b)** the case of a stationary observer and source moving at low altitude $d = 10^4\,\mathrm{m}$ and with relativistic velocity $\mathrm{v} = 0.6 \times c$.

4. Conclusions

The Doppler shift formulas derived without making the usual assumption of “plane wave” (“very high” frequency) clearly show the non-linear character of the Doppler factor. As a result, the Doppler

Effect differently shifts the different frequencies present in the electromagnetic radiation. That fact could be of importance in Astronomy. In the case of the longitudinal Doppler Effect ($\theta = 0^0$ or $\theta = 180^0$) and in the case of the transversal Doppler Effect the different frequencies are equally shifted. In the case of oblique incidence, the Doppler Effect shifts more the lower frequencies.

References

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