A new instantaneous frequency estimation method based on the use of image processing techniques

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ABSTRACT

The aim of this paper is to present a new method for the estimation of the instantaneous frequency of a frequency modulated signal, corrupted by additive noise. This method represents an example of fusion of two theories: the time-frequency representations and the mathematical morphology. Any time-frequency representation of a useful signal is concentrated around its instantaneous frequency law and realizes the diffusion of the noise that perturbs the useful signal in the time-frequency plane. In this paper a new time-frequency representation, useful for the estimation of the instantaneous frequency, is proposed. This time-frequency representation is the product of two others time-frequency representations: the Wigner-Ville time-frequency representation and a new one obtained by filtering with a hard thresholding filter the Gabor representation of the signal to be processed. Using the image of this new time-frequency representation the instantaneous frequency of the useful signal can be extracted with the aid of some mathematical morphology operators: the conversion in binary form, the dilation and the skeleton. The simulations of the proposed method have proved its qualities. It is better than other estimation methods, like those based on the use of adaptive notch filters.

Keywords: instantaneous frequency, time-frequency representation, mathematical morphology, images, operators.

1. INTRODUCTION

The signals analyzed in this paper have low signal to noise ratios. The estimation of the instantaneous frequency of a mono component signal, not corrupted by noise, is a problem already studied\textsuperscript{1,2}. When the useful signal is multi component or it is perturbed by additive noise, the estimation problem is more complicated and the algorithms already reported generally don’t work. This is the reason why we propose here a method based on the use of time-frequency representations. These distributions have two useful properties:

1. They have a very good concentration around the curve of the instantaneous frequency of the analyzed signal\textsuperscript{3};
2. They realize a diffusion of the perturbation noise’s power in the time-frequency plane.

So, computing the time-frequency representation of the analyzed signal: \( x(t) = s(t) + n(t) \), where \( n(t) \) is the perturbation, we can obtain a good estimation of the ridges of the time-frequency representation of the signal \( s(t) \). Projecting these ridges on the time-frequency plane we obtain a good estimation of the instantaneous frequency of the signal \( s(t) \). There are many methods to estimate the ridges of a time-frequency representation\textsuperscript{4}. We propose here a new method, based on the use of mathematical morphology operators: the conversion in binary form, the dilation and the skeleton. The simulations of the proposed method have proved its qualities. It is better than other estimation methods, like those based on the use of adaptive notch filters.

2. THE ROLE OF THE TIME-FREQUENCY REPRESENTATIONS

The role of the time-frequency representations in our estimation method is to spread the noise in the time-frequency plane and to locate the instantaneous frequency of the useful signal. There are a lot of time-frequency representations:
the Short-time Fourier transform, the wavelet transform (linear representations) and the members of the Cohen class (bilinear representations). The definition of a linear time-frequency representation is:

If the following conditions are satisfied:

1) \( A \subset \mathbb{R}^n \), \( K : \mathbb{R} \times A \to \mathbb{C} \);

2) \( (\forall) a \in A, \quad \tau \mapsto K(\tau,a) \) is measurable and \( \int_{-\infty}^{\infty} ||K(\tau,a)||^2 d\tau = 1 \);

3) \( (\forall) \omega \in \mathbb{R}, \quad a \mapsto \mathcal{F}[K(\tau,a)](\omega) \) is measurable and \( \int_{A} |\mathcal{F}[K(\tau,a)](\omega)|^2 da = C < \infty \);

then the function:

\[
\text{TF}_x : A \times \mathbb{R} \to \mathbb{C}, \quad \text{TF}_x(t, \omega) = \langle x(\tau), K(\tau - t, \omega) \rangle = \int_{-\infty}^{\infty} x(\tau) K^*(\tau - t, \omega) d\tau
\]

is named linear time-frequency representation of the finite energy signal \( x(\tau) \).

The two variables function \( K(u, v) \) is the kernel of the linear time-frequency representation. For different kernels different time-frequency representations are obtained. The kernel for the Short-time Fourier transform is \( K_{\text{STFT}}(\tau, a) = w(\tau) e^{-j\omega \tau} \). If the window, \( w(\tau) \), is gaussian then the corresponding time-frequency representation is the Gabor transform. The kernel for the Continuous Wavelet Transform is \( K_{\text{CWT}}(\tau, a) = \sqrt{a} \psi\left(\frac{\tau}{a}\right) \), where the function \( \psi \) is a mother of wavelets.

The bilinear time-frequency representations of the Cohen’s class can be computed for the signal \( x \) with the relation:

\[
c^{TF}_x(t, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(s + \frac{\tau}{2}) \ast x(s - \frac{\tau}{2}) e^{-j(\omega s - a\omega + \omega \tau)} f(u, \tau) dud\tau
\]

where \( f(u, \tau) \) is another kernel. For the Wigner-Ville time-frequency representation this kernel is unitary.

Some of the members of the Cohen’s class realize a good localization of the ridges of the analyzed signal. A very good example is the Wigner-Ville distribution. For the mono component signal not perturbed by noise:

\[
s(t) = \cos(2\pi \tau^2)
\]

the associated analytical signal is:

\[
s_a(t) = e^{-j2\pi t^2}
\]

and the instantaneous frequency is:

\[
f_i(t) = 2t
\]

The Wigner-Ville representation of the signal \( s_a(t) \) is:
\[
\text{TF}_{\text{W-V}}^W(t, \omega) = 2\pi \cdot \delta(\omega - 4\pi)
\]

So, this time-frequency representation is perfectly concentrated on the curve \( \omega = 2\pi \cdot f(t) \) of the instantaneous frequency of the signal \( s(t) \). Hence for the estimation of the instantaneous frequency of the signal \( s(t) \) the better time-frequency representation is the Wigner-Ville distribution.

But for the instantaneous frequency estimation of multi component signals or of signals perturbed by noise, the linear time-frequency representations are more useful due to the absence of the interference terms specifics for the bilinear time-frequency representations. The good concentration around the instantaneous frequency law properties of the linear time-frequency representations of signals with double modulation of the form:

\[
s(t) = A(t) \cdot e^{j\varphi(t)}
\]

Is recognized² that the Gabor time-frequency representation realizes the better localization in the time-frequency plane (see the Heisenberg principle).

### 2.1. Spreading the noise in the time-frequency plane

To observe this effect a statistical analysis of time-frequency representations is done. We suppose that the signal to be represented in the time-frequency plan, \( n(t) \), is a stationary noise.

#### 2.1.1. The case of linear time-frequency representations

The linear time-frequency representation of the noise \( n(t) \) is:

\[
i\text{TF}_n(t, \omega) = \int n(\tau) K^*(\tau - t, \omega) d\tau
\]

(7)

The system for the computation of this time-frequency representation is a time invariant linear system with the impulse response \( K^*(-t, \omega) \), where the frequency, \( \omega \), represents a parameter. At any frequency, \( \omega \), this system responds to the input signal, \( n(t) \), with the signal \( n_\omega(t) \), the linear time-frequency representation, \( i\text{TF}_n(t, \omega) \), computed at that frequency.

This is a random process with the mean:

\[
E[i\text{TF}_n(t, \omega)] = M_n \int K^*(\tau - t, \omega) d\tau
\]

(8)

where \( M_n \) represents the mean of the noise \( n(t) \). If this noise has a zero mean then the mean of its linear time-frequency representation is also null. The correlation function of the time-frequency representation from the relation (7) is:

\[
E[i\text{TF}_n(t_1, \omega_1) i\text{TF}_n(t_2, \omega_2)] = \int \int K^*(\tau_1 - t_1, \omega_1) K^*(\tau_2 - t_2, \omega_2) R_n(\tau_1 - \tau_2) d\tau_1 d\tau_2
\]

(9)
where \( R_n(\tau) \) represents the correlation of the noise \( n(t) \). For a zero mean white noise with standard deviation \( \sigma \) the last relation becomes:

\[
E\{TF_n(t_1, \omega_1) | TF_n(t_2, \omega_2)\} = \sigma^2 \cdot \int K^*(\tau_1 - t_1, \omega_1) K^*(\tau_1 - t_2, \omega_2) d\tau_1 = R_{TF_n}(t_1 - t_2, \omega_1 - \omega_2)
\]  

(10)

So, any linear time-frequency representation correlates the input noise. This correlation can be avoided only for discrete linear time-frequency representations. It is well known the whitening effect of the discrete wavelet transform.

The power of the output signal of the system in figure 1 is:

\[
E_{\text{no}} = \frac{1}{2\pi} \int \left| N_{\text{o}}(\omega) \right|^2 d\omega = \frac{1}{2\pi} \int \left| N(\omega) \right|^2 \left| \mathcal{F}\left\{ K^{-1}(t, \omega) \right\}(\omega) \right|^2 d\omega \leq \frac{1}{2\pi} \int \left| N(\omega) \right|^2 d\omega = E_n
\]  

(11)

So, at any frequency, the power of the signal \( n_0(t) \) is inferior to the power of the signal \( n(t) \). Hence, the linear time-frequency representation realizes a spreading of the noise in the time-frequency plane.

2.1.2 The case of bilinear time-frequency representations

In the following a statistical analysis of bilinear time-frequency representations is presented. A Cohen’s class time-frequency representation of the signal \( n(t) \) can be computed using the relation (1). Its mean is:

\[
E\{TF_n(t, \omega)\} = \frac{1}{2\pi} \int \int \int R_n(\tau) e^{-j(\omega \tau - \xi + \mu)} f(u, \tau) du d\xi d\tau
\]  

(12)

If \( n(t) \) is a zero mean white noise with standard deviation \( \sigma \) then the mean of its time-frequency representation becomes:

\[
E\{TF_n(t, \omega)\} = \sigma^2 \cdot f(0,0)
\]  

(13)

The Cohen’s class time-frequency representations are not zero mean random processes. The expectation of the Wigner-Ville time-frequency representation for a zero mean white noise with standard deviation \( \sigma \) is equal with the power of this noise, \( \sigma^2 \). So, a very good estimation of the input noise’s power can be realized by statistical averaging of few Wigner–Ville time-frequency representations of different realizations of this noise. The computation of the correlation of a Cohen’s class time-frequency representation for the signal \( n(t) \) is more difficult.
where \( 4R_n(\tau_1, \tau_2, s_1, s_2) \) represents the fourth order correlation of the input signal. This function can be computed with the relation:

\[
4R_n(\tau_1, \tau_2, t_1, t_2) = \text{cum}_4\{n(t_1) \cdot n(t_2) \cdot n(t_3) \cdot n(t_4)\} + R_n(t_1, t_2) \cdot R_n(t_3, t_4) + R_n(t_1, t_3) \cdot R_n(t_2, t_4) + R_n(t_1, t_4) \cdot R_n(t_2, t_3)
\]

where the forth order cumulant can be computed using the relation:

\[
\text{cum}_4\{n(t_1) \cdot n(t_2) \cdot n(t_3) \cdot n(t_4)\} = E\{n^4\} - 3E\{n^2\}^2
\]

If \( n(t) \) is a zero mean white noise with standard deviation \( \sigma \) then:

\[
\text{cum}_4\{n(t_1) \cdot n(t_2) \cdot n(t_3) \cdot n(t_4)\} = 0, \quad \text{and} \quad 4R_n(t_1, t_2, t_3, t_4) = \sigma^4(\delta(t_1 - t_2)\delta(t_3 - t_4) + \delta(t_1 - t_3)\delta(t_2 - t_4) + \delta(t_1 - t_4)\delta(t_2 - t_3))
\]

and the correlation of its Cohen’s class time-frequency representation becomes:

\[
E\{C_{\text{TF}_n}(t_1, \omega_1) \cdot C_{\text{TF}_n}(t_2, \omega_2)\} = -\frac{\sigma^4}{2\pi} \cdot \mathcal{F}_2[f(u_2, \tau_2)\delta(-u_2, \tau_1)(\omega_1 + \omega_2, t_1 + t_2) + \sigma^4f(0,0) + \frac{\sigma^4}{\pi} \cdot \mathcal{F}_2[f(u_2, \tau_2)\delta(-u_2, \tau_2)(\omega_2 - \omega_1, t_2 - t_1)]
\]

where \( \mathcal{F}_2 \) represents the two-dimensional Fourier transform. For the case of the Wigner – Ville time-frequency distribution, the last relation becomes:

\[
E\{\text{TF}_n^{W-V}(t_1, \omega_1) \cdot \text{TF}_n^{W-V}(t_2, \omega_2)\} = 4\pi\sigma^4 \cdot \delta(\omega_2 - \omega_1) \cdot \delta(t_2 - t_1) + \sigma^4 - 2\pi\sigma^4 \cdot \delta(\omega_2 + \omega_1) \cdot \delta(t_2 + t_1)
\]

So, the Wigner – Ville time-frequency representation of a zero mean white noise with standard deviation \( \sigma \) is a two dimensional random process, very close to a two dimensional white noise. So, generally, the Cohen’s class time-frequency representations correlates the input signal (see the relation (18)) but the Wigner –Ville time-frequency representation is an exception. Because the Wigner – Ville distribution is a spectro-temporal density of energy that don’t correlates the input noise, it posses the noise power’s spreading in the time-frequency plane effect, that represents
the subject of this paragraph. Those are the reasons why this time-frequency representation was selected for the method presented in this paper.

2.2. A new time-frequency representation

To combine the advantages of the Gabor time-frequency representation (the absence of interference terms) and of the Wigner-Ville time-frequency representation (the very good concentration in the time-frequency plane) the following algorithm can be used:

1. The computation of the Gabor transform of the signal \( x(t) \), \( \text{TF}_G^x(t, \omega) \). The noise’s power is diffused in the time frequency plane. Only a small part of this power affects the ridges of \( \text{TF}_G^s(t, \omega) \).

2. The filtering of the image obtained, \( y(t, \omega) = \text{TF}_G^x(t, \omega) \), with the aid of a hard-thresholding filter. This system has the following input-output relation:

\[
    z(t, \omega) = \begin{cases} 
        y(t, \omega) & \text{if } |y(t, \omega)| \geq \tau, \\
        0 & \text{if } |y(t, \omega)| < \tau
    \end{cases}
\]  

(20)

where \( \tau \) is a threshold. This threshold’s value is selected to be equal with a fraction of the maximum value of the modulus of \( \text{TF}_G^x(t, \omega) \). Doing so, a prototype time-frequency representation, \( z(t, \omega) \), the de-noised version of the time-frequency representation \( y(t, \omega) \), is obtained. The de-noising operation decreases the amount of noise that perturbs the ridges of \( \text{TF}_G^s(t, \omega) \) and brings to zero the values in the rest of the time-frequency plane. This is the reason why the interference terms of the bilinear time-frequency representation, that will be used in the following step, will be eliminated in the final step of this method.

3. The computation of the Wigner – Ville time-frequency representation of the signal \( x(t) \), \( \text{TF}_V^x(t, \omega) \).

The goal of this step is to enhance the localization on the curve of instantaneous frequency of the signal \( s(t) \), in the time-frequency plane.

4. The new time-frequency representation, \( \text{TF}_X^{\text{new}}(t, \omega) \) that represents the subject of this paragraph, is computed by the multiplication of the modulus of the prototype time-frequency representation, \( z(t, \omega) \) with \( \text{TF}_V^x(t, \omega) \). The effect of this multiplication is the cancellation of the interference terms of the Wigner – Ville distribution and the very good localization of the ridges of the result in the time-frequency plane.

3. THE ROLE OF THE MATHEMATICAL MORPHOLOGY

Some mathematical morphology operators can be used to estimate the ridges of the time-frequency representation. There are two goals of this estimation procedure:

- to de-noise the time-frequency representation;
- to extract the projection of the ridges of the time-frequency representation on the time-frequency plane.

In the following is presented a list of mathematical morphology operators useful for the purpose of this paper.

- The conversion in the binary form. This operator realizes a thresholding of the time-frequency representation image. In fact this is a new de-noising procedure. So, the effect of the use of this operator is a de-noising of the image of the time-frequency representation.
- The R-h-maxima operator. Using this operator the peak regions of a time-frequency representation are extracted. So, this operator is useful for the detection of the ridges of a time-frequency representation.
- The exclusive or operator. Taking into account the random nature of a time-frequency representation of a random process, the use of a Boolean operator having for entries two slightly different versions of the image of the
time-frequency representation (the image obtained after the conversion in the binary form and the image obtained after the application of the R-h-maxima operator) has a de-noising effect.

- The skeleton operator\(^7\). Using this operator an estimate of the projection of the ridges of the time-frequency representation on the time-frequency plane can be obtained.

### 4. THE NEW ALGORITHM

The algorithm that represents the aim of this paper has the following steps:

1. The new time-frequency representation of the signal \(x(t)\) is computed.
2. Its image is converted in binary form.
3. A dilatation of the image obtained is performed obtaining a new image. (This dilatation is realized to prevent the connectivity’s loose of the instantaneous frequency).
4. Applying the skeleton operator to the last image an estimation of the instantaneous frequency of the signal \(s(t)\) is obtained. This image represents the result of the proposed estimation method.

### 5. SIMULATION RESULTS

In the following is presented a simulation result of the application of the new algorithm. The signal \(s(t)\) is a mono component signal. Its modulation law is quadratic. It has constant amplitude. The perturbation \(n(t)\) is a train of noise pulses. This kind of perturbation appears frequently in practice. The value of the signal to noise ratio (SNR) of the analyzed signal is of 1.34. The modulus of the Gabor transform of the signal \(x(t)\) is represented in figure 1. After the filtering with the hard-thresholding filter is obtained a new image with the modulus presented in figure 2. The image of the absolute value of the Wigner – Ville time-frequency representation of the considered signal is presented in figure 3. The modulus of the new time-frequency representation of the considered signal is presented in figure 4. After the conversion of this image in binary form the result presented in figure 5 is obtained. After the dilation of this image the result presented in figure 6 is obtained. Applying the skeleton operator, the image in figure 7, the result of the simulation method is obtained.

![Figure 1. The modulus of the Gabor transform.](image-url)
Figure 2. The de-noised Gabor transform.

Figure 3. The modulus of the Wigner–Ville transform.

Figure 4. The modulus of the new representation.

Figure 5. The image obtained after conversion in binary form.
Comparing the real instantaneous time-frequency representation of the signal \( s(t) \) with the result obtained in figure 7, the estimation error can be appreciated. For this example the maximum error appears at the frequency 3748 Hz and at the moment 0.464 s having an absolute value of 142 Hz. So the maximum relative error is of 0.0284.

6. CONCLUSIONS

The instantaneous frequency estimation method proposed in this paper has performances similar or better than other methods\(^4\), \(^8\), \(^9\), \(^10\), \(^11\).

This is a complete method, after the acquisition of the signal \( x(t) \), the plot of the instantaneous frequency of the signal \( s(t) \) is directly obtained.

The method is quite universal, the SNR of the input signal can be very small and the result is not affected by the statistics of the perturbation \( n(t) \). For example similar results can be obtained for white Gaussian noise.

For signals, \( s(t) \), with constant amplitude, using the method proposed in this paper, the reconstruction can be also achieved. So, this method can be regarded like a de-noising method for frequency modulated signals with constant amplitude. Knowing the instantaneous frequency law, the frequency-modulated signal with constant amplitude can be synthesized very easy. For the case of the use of the continuous wavelet transform a similar conclusion is reported \(^8\).

Such a method outperforms the majority of de-noising methods for the frequency-modulated signals being able to reconstruct very low SNR signals. Other SNR enhancement methods, like for example the Donoho’s de-noising method\(^{12}\), are not designed for the treatment of low SNR signals.

The multiplication of a linear and a bilinear time-frequency representation conducts to an important reduction of the interference terms (compare for example the figures 3 and 4). This interference terms reduction method has similar results with the reallocation method\(^{13}\), but is simpler.

The class of morphological operators used in the proposed estimation method can be extended.

Another future research direction is the complete statistical analysis of the proposed method. The algorithm proposed in this paper is of empirical nature. We have not studied yet the optimization of the selection methods for the parameters required in every step of the implementation.

The aim of this paper is only to propose an alternative method for the instantaneous frequency estimation, based on the conjoint use of two very modern theories, that of time-frequency representations and that of mathematical morphology. This connection is very important because the time-frequency representations are generally used for the processing of signals with only one dimension and the mathematical morphology is used to process images. Our proposition permits
to use the image processing techniques to the analysis of monodimensional signals. This strategy permits the
enhancement of the set of signal processing methods with the aid of some methods developed in the context of image
processing. This contribution of the image processing theory to the development of the signal processing theory is very
important taking into account the fact that at the basis of the development of the image processing theory lies the
signal processing theory.

The estimation method proposed in this paper can be used in a lot of applications. Some of them, like radar, sonar, or
telecommunications are already recognized as applications of the time-frequency representation theory.

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