ON THE INITIALIZATION ERRORS OF THE DISCRETE WAVELET TRANSFORM ALGORITHM

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ABSTRACT
The discrete wavelet transform algorithm can be used to process (to compress or to de-noise for example) continuous in time signals. To use this processing method, the initialization errors of the discrete wavelet transform algorithm must be minimized. This is the aim of this paper. In fact in this paper are put together the principal results obtained in the wavelet's literature concerning the initialization of the discrete wavelet transform. These results are justified and are presented in a unitary manner. We give a strategy to accomplish this minimization. Some examples are presented. In this paper are estimated for the first time the errors occurring in the initialization process. A superior bound of these errors is presented too.

1. INTRODUCTION
A modern problem in signal processing theory is the analysis of non-stationary signals. The tools for this analysis are the time-frequency representations. One of the most important time-frequency representation is the Continuous Wavelet Transform (C.W.T), [1].

Given a non-stationary signal \( x(t) \), wavelet transform consists of computing coefficients that are inner products of the signal and a family of "wavelets". The wavelet corresponding to scale \( a \) and time-location \( b \) is:

\[
\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left( \frac{t-b}{a} \right)
\]

where \( \psi(t) \) is the "mother of wavelets". The CWT of \( x(t) \) is:

\[
CWT_x(a,b) = \int_{-\infty}^{\infty} x(t)\psi_{a,b}^{*}(t)dt
\]

Time \( t \) and time-scale parameters vary continuously. This transform can be discretized. If the time remains continuous but time-scale parameters \( (b,a) \) are sampled on a "dyadic" grid, then the wavelet series coefficients are obtained:

\[
C_{j,k} = CWT_x(2^j,k2^j)
\]

The approximation of \( x(t) \) by \( x_j(t) \) is realized with the error \( e_j(t) = x(t) - x_j(t) \). For every multiresolution analysis of \( L^2(R) \), \( V_j \) \( j \in \mathbb{Z} \), a correspondent orthogonal decomposition of \( L^2(R) \), \( W_j \) \( j \in \mathbb{Z} \) can be built. The error \( e_j(t) \) is the projection of the signal \( x(t) \) on the space \( W_j \). It can be written:

\[
x_0(t) = x_j(t) + \sum_{j=1}^{J} e_j(t)
\]

The coefficients \( c_{j,k} = \langle \psi_j(t), \psi_{j,k}(t) \rangle \) can be computed with the following recurrence relation:

\[
c_{j,k} = \sum_{l} b_{j-1,l} g_{2n-l}^{*}
\]

where:

\[
h_{2n-l} = \langle \Phi_{1,n}(t), \phi(t-l) \rangle
\]

So, using the coefficients \( b_{0,n} \) of the signal \( x_0(t) \), the \( x_j(t) \) and \( e_j(t) \), \( j=1,...,J \) can be obtained. The transformation of the discrete signal \( b_{0,k} \) in the sequence of signals: \( c_{1,k} \), \( c_{2,k} \), \( ...c_{J,k} \), \( b_{J,k} \) is named Discrete Wavelet Transform (DWT). The main problem of this paper is to obtain the signal \( x_0(t) \) (its coefficients \( b_{0,n} \)) from a known signal \( x(t) \). This operation represents the initialization of the DWT. The signal \( x(t) \) can be exactly reconstructed after the application of the DWT and the inverse DWT to the sequence \( b_{0,k} \). Because this signal represents the projection of the signal \( x(t) \) on the space \( V_0 \), it can be computed using a projection filter. This is the reason why, in the reconstruction phase, the signal \( x(t) \) can be generated using the signal \( x_0(t) \) (that can be obtained after the application of the inverse DWT) and the inverse system corresponding to the projection filter.
2. THE INITIALIZATION PROBLEM

For the use of an algorithm to the processing of a continuous in time signal, \( x(t) \), this signal must be sampled, obtaining the sequence \( x[n] \). The initialization of the DWT consists in the computation of the sequence \( b_{0,k} \) starting from the sequence \( x[n] \). The DWT is a very fast transformation. It is faster then the FFT. The subject of this paper is to obtain a good approximation of the sequence \( b_{0,n} \), starting from the sequence \( x[n] \). The problem is to find the impulse response of a discrete in time system, \( \alpha[n] \), such that it’s response \( y[n] \), to the input signal \( x[n] \), to be a good approximation of the signal \( b_{0,n} \). This is a very interesting problem, pointed out in [3], [4], [6] and [7].

2.1 How to obtain the sequence \( b_{0,n} \) starting from the signal \( x(t) \)?

The system in figure 1 transforms the signal \( x(t) \) into the sequence \( b_{0,n} \).

![Figure 1. The transformation of the signal \( x(t) \) into the sequence \( b_{0,n} \).](image)

The expression of the signal \( u(t) \), in figure 1, is:

\[
u(t) = x(t) \ast \phi^{\ast*}(t) = \int_{-\infty}^{\infty} x(\tau) \phi^{\ast*}(t-\tau) d\tau\]

Sampling this signal we obtain:

\[y(t) = \sum_{k=-\infty}^{\infty} u(k) \delta(t-k)\]

The samples of \( u(t) \) have the values:

\[u(k) = \int_{-\infty}^{\infty} x(\tau) \phi^{\ast*}(\tau-k) d\tau = <x(\tau), \phi(\tau-k)>\]

So:

\[u(k) = b_{0,k}\]

Hence the system in figure 1 transforms the signal \( x(t) \) into the sequence \( b_{0,k} \).

2.2 The Computation of the Impulse Response \( \alpha[n] \)

The initialization problem is presented in figure 2. The condition to cancel the error \( e[n] \) is:

\[F_d(x[n]) F_d(\alpha[n]) = F_d((x(t) \ast \phi^{\ast*}(t)) |_{t=n})\]

where \( F_d \) represents the Fourier transform in discrete time. So the expression of the impulse response \( \alpha[n] \) is:

\[
\alpha[n] = F_d^{-1} \left( \frac{F_d((x(t) \ast \phi^{\ast*}(t)) |_{t=n})}{F_d(x[n])} \right) (4)
\]

Shensa proposed this exact solution of the DWT initialization problem, too, without derivation, in [3], (relation (4.12.c)). Unfortunately this relation can not be applied when the expression of the signal \( x(t) \) is unknown. The continuous in time convolution requested in (4) can not be computed with a computer. In the following, two particularization of (4), more useful in practice, are presented.

- **Case I**

  When the signal \( x(t) \) is band-limited, \( x(t) \in B_{p}^{2} \), the relation (4) becomes:

  \[
  \alpha[n] = z[n]
  \]  

  The Fourier transform of the signal \( z[n] \) is:

  \[
  F_d(z[n]) = F_d(\phi^{\ast*}(t) \psi(t)) |_{t=n, \omega = \Omega}
  \]

  where \( F_d \) represents the Fourier transform for signals continuous in time. This is so because the continuous in time signal with the Fourier transform \( F_d(\phi^{\ast*}(t)/\omega) \mid_{\Omega} \) is a band-limited one, like \( x(t) \). This is the reason why their continuous convolution is a member of the \( B_{p}^{2} \) space too. But, for such
signals, the Fourier transform in time discrete of their sampled version (obtained using unitary step), is identical in the interval $[-\pi, \pi]$ with their Fourier transform in continuous time.

- **Case II.** To the hypothesis of case I is added the supplementary hypothesis that $\Phi^v(t) \in B_{p^2}$. In this case the relation (4) becomes (see the relation (5) and the case I):

$$\alpha_a[n] = \Phi^v[n]$$

(6)

So, the system while the impulse response $\alpha[n]$ is equivalent with the system with impulse response $\Phi^v(t)$, on the base of the impulse invariance method, [5]. This approximation of the solution of the initialization problem, without derivation, is proposed in [6] (relation (15) for $\chi(t)=\delta(t)$) and in [7] (relation (14)).

### 3. THE ESTIMATION OF THE APPROXIMATION ERROR

At the beginning are presented some preliminary results.

**P1.** The discrete in time signal obtained by uniform sampling with a unitary step of a finite energy signal has finite energy.

**P2** The convolution of two finite energy signals continuous in time is a new finite energy signal.

**P3** The convolution of two finite energy signals discrete in time is a new finite energy signal discrete in time.

Now, the approximation errors for the initialization methods in relation (5) and (6) can be estimated. The initialization error is:

$$e[n] = u[n] \cdot v[n] \text{ where } u[n] = (x(t) * \Phi^v(t))_{|_{t=n}}$$

$$v[n] = x[n] * \alpha_a[n] = \Phi^v[n]$$

(7)

A superior bound of this error can be estimated too. Indeed:

$$|e[n]| \leq |x(\tau) \cdot \Phi^v(n-\tau) >_{L^2} | + |x[k] \alpha_a[n-k] >_{L^2}$$

(8)

Because there is an isomorphism between the spaces $B_{p^2}$ and $L^2$ introduced by sampling, we can observe that the error $e[n]$ is zero when the functions $x(t)$ and $\Phi(t)$ are elements of $B_{p^2}$. A different strategy for the decreasing of the initialization error was considered in [4]. This strategy is based on adaptive filtering. Its advantage is the fact that the analytical expression of the corresponding scaling function is not requested. The disadvantage is the amount of computation requested by the adaptive filtering procedure. Unfortunately the authors of this article have not derived a superior bound for the initialization error obtained using their strategy. In the following we present some examples to compare the precision of different initialization methods.

### 4. SOME EXAMPLES

**A. Example 1**

Let: $$x(t) = \text{sinc}(\pi t)$$

Because this is a band-limited signal, the appropriate scaling function for the computation of its discrete wavelet transform is:

$$\Phi(t) = \text{sinc}(\pi t)$$

because this is a band-limited signal too. Using the relation (4) it can be written:

$$\alpha[n] = \delta[n]$$

Hence:

$$b_{0,n} = \text{sinc}(\pi n) = \delta[n]$$

Using the relation (7) the initialization error can be computed. It is equal with zero at any moment. So the initialization procedure is exact in this case.

For this example the other initialization methods (described in the relations (5) and (6)) are exact too. Unfortunately the scaling function chosen in this example has not a compact support.

**B. Example 2**

Let:

$$x(t) = (\sigma(t - \frac{9}{10}) - \sigma(t - \frac{11}{10}))$$

Tacking into account the waveform of this signal, the appropriate wavelet for its analysis, is the Haar wavelet. So:

$$\Phi(t) = \sigma(t) - \sigma(t - 1)$$

Computing the convolution in (4) we obtain:

$$x(t) * \text{sinc}(\pi t)$$

But:

$$\alpha[n] = \delta[n - 1]$$

So, using (4) we obtain:

$$x[n] = \frac{1}{10} \delta[n - 1] + \frac{1}{10} \delta[n - 2]$$

and:

$$b_{0,n} = x[n] * \alpha[n] = \frac{1}{10} (\delta[n - 2] + \delta[n - 3])$$

The initialization proposed in (4) is not exact because in this example the function $\Phi(t)$ is not band-limited (the initialization error can be computed using the relation (7)) and the initialization proposed in (6) can not be applied because the function $\Phi(t)$ can not be sampled (this is not a continuous function).
5. CONCLUSION
The use of DWT is a very elegant solution for the digital signal processing of a signal continuous in time. In this purpose it must be initialized. The initial sequence \(b_0 n\) represents the result of the convolution between the sampled version of the analyzed signal \(x[n]\) and an impulse response \(a[n]\). The exact expression of this impulse response is presented in the relation (4). Unfortunately is difficult to apply this solution in practice, because the convolution of two continuous in time signals can not be exactly computed with numerical algorithms and because there are occasions when one or both expressions of the signal \(x(t)\) and of the scaling function are not known. This is the reason why approximations for the expression of the impulse response are presented too, in the relation (5) and (6). The expression in the relation (6) suppose the construction of the initialization filter with the aid of a method for the equivalence of a continuous in time system with a discrete in time one. In fact the equivalence based on the invariance of the impulse response is used. Of course there are other equivalence method too. So other initialization filters, similar to the one presented in the relation (6) can be built using the equivalence method based on the approximation of a differential equation with an equation with finite differences or using the equivalence method based on the bilinear transform. The expression of the impulse response \(a[n]\) depends on the scaling function \(\phi(t)\) that generates the specified space \(V_0\), too. This is the reason why we have chosen in the examples presented the most appropriate scaling functions. Unfortunately there are some wavelet’s mothers, with corresponding scaling functions, without analytical expression (this is the case for the compact support wavelets introduced by Ingrid Daubechies). In this paper are estimated for the first time the errors occurring in the initialization process when the relation (5) and (6) are used. A superior bound of these errors is presented too.

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Other useful references