Capacity of Frequency Selective Channels.
Part I. Time Invariant Channels.

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Abstract. There are presented some relations to compute the frequency channel capacity in a form that can be used for a class of Rayleigh channels. The capacity loss of the frequency selective channels comparative with flat channels capacity is analyzed.

Keywords: channel capacity, frequency selective channel, Rayleigh channel

1. INTRODUCTION

A frequency selective communication channel can be modeled like in fig.1. The signal transmitted over the channel, \(x(t)\), has the power spectral density, psd, \(S_x(f)\). The additive noise, \(n(t)\), affecting the transmission is considered to be a zero-mean, Gaussian, with psd \(N(f)\). The channel frequency bandwidth is \(W\). If the noise is white, his psd is constant over the channel bandwidth, namely \(N(f)=N_0\). When the noise is colored, his psd \(N(f)\) is not flat in the channel bandwidth. The output signal \(y(t)\) has the psd \(S_y(f)\) which can be calculated with:

\[ S_y(f) = S_x(f) |C(f)|^2 + N(f) \] \hspace{1cm} (1)

Here \(C(f)\) is the frequency response of the channel. From (1) we obtain:

\[ S_y(f) = \frac{S_x(f) + N(f)}{|C(f)|^2} |C(f)|^2 \] \hspace{1cm} (2)

and using the notation

\[ Z(f) = \frac{N(f)}{|C(f)|^2} \] \hspace{1cm} (3)

it becomes

\[ S_y(f) = [S_x(f) + Z(f)] |C(f)|^2 \] \hspace{1cm} (4)

The function \(Z(f)\) is the corresponding psd of the noise \(n(t)\) reflected at the channel input, like a noise \(z(t)\) (see fig.2) \[ |C(f)|^2 \] and \(Z(f)\) for a frequency selective channel are presented in fig.3 and so is the psd \(S_y(f)\).

![Figure 1. The frequency selective channel model](image1)

is white, his psd is constant over the channel bandwidth, namely \(N(f)=N_0\). When the noise is colored, his psd \(N(f)\) is not flat in the channel bandwidth. The output signal \(y(t)\) has the psd \(S_y(f)\) which can be calculated with:

![Figure 2. The frequency selective channel with noise reflected at the channel input](image2)
2. Capacity of the frequency selective channel

We will consider a frequency bandwidth \([f_k, f_k + \Delta f]\), where \(\Delta f\) is small enough to write:

\[
|C(f)|^2 \approx |C(f_k)|^2 \quad (5)
\]

For this channel, the received signal power is

\[
S_x(f_k) |C(f_k)|^2 \Delta f,
\]

and the noise power is

\[
Z(f_k) |C(f_k)|^2 \Delta f = N(f_k) \Delta f.
\]

Its capacity, \(C_k\), can be calculated with:

\[
C_k = \Delta f \cdot \log_2 \left(1 + \frac{S_x(f_k)}{Z(f_k)}\right) \quad (6)
\]

If we split now the bandwidth \(W\) of the channel in \(N\) adjacent subbands, each having \(\Delta f = W/N\) bandwidth, the relations (5) are true for each subband. Then, eq.(6) is also true for each subband and the frequency selective channel capacity \(C\) can be approximated by:

\[
C = \sum_{k=1}^{N} \Delta f \cdot \log_2 \left[1 + \frac{S_x(f_k)}{Z(f_k)}\right] \quad (7)
\]

If \(\Delta f \to 0\), that means \(N \to \infty\), the sum of eq.(7) tends to a definite integral:

\[
C = \int \log_2 \left[1 + \frac{S_x(f)}{Z(f)}\right] \cdot df \quad (8)
\]

or:

\[
C = \int \log_2 \left[1 + \frac{S_x(f) |C(f)|^2}{N(f)}\right] \cdot df \quad (9)
\]

The relations (8) or (9) give us the frequency selective channel capacity.

2.1 Capacity of the channels with optimal power distribution affected by colored noise

The question is how to distribute the psd of the transmitted signal, \(S_x(f)\), to obtain \(C_{\text{max}}\), the maximum capacity of the frequency selective channel. The power repartition is done by the water-filling theorem. We must optimize \(C\) from the eq. (7), with two constraints. First is that the transmitted power, \(P_x\), is a constant, but it must be distributed in a different way on the bandwidth \(W\):

\[
P_x = \sum_{k=1}^{N} S_x(f_k) \cdot \Delta f \quad (10)
\]

where \(S_x(f)\) must be considered only for \(f \geq 0\). The second constraint results because \(S_x(f) \geq 0\) and it is:

\[
S_x(f_k) \geq 0 \quad \forall k \in \{1, N\} \quad (11)
\]

It can be shown (see annexe A.1) that the power repartition must be:

\[
S_x(f) = \begin{cases} 
\Lambda - Z(f) & \text{if } Z(f_k) < \Lambda \\
0 & \text{if } Z(f_k) > \Lambda 
\end{cases} \quad (12)
\]

The threshold \(\Lambda\) is determined from eq.(10). We observe that the power is not distributed over all the \(N\) subchannels but only over \(M\) channels, \(M \leq N\), where \(Z(f_k) < \Lambda\). Applying eq. (10) for the \(M\) subbands that receive all the power, \(P_x\), it can be obtained the threshold \(\Lambda\). So, the power can be computed with:

\[
P_x = \sum_{k=1}^{M} [\Lambda - Z(f_k)] \cdot \Delta f \quad (13)
\]

where \(\langle M \rangle\) is the set of values \(k\) for which \(S_x(f_k) > 0\). They defined an effective occupied bandwidth \(W_{\text{eff}} = M \Delta f\) and \(W_{\text{eff}} \leq W\).

For \(\Delta f \to 0\) or \(N \to \infty\) and \(M \to \infty\), we have:

\[
S_x(f) = \begin{cases} 
\Lambda - Z(f) & \text{if } Z(f) < \Lambda \\
0 & \text{if } Z(f) > \Lambda 
\end{cases} \quad (14)
\]

and, from (13) it results:

\[
P_x = \Lambda \int_{W_{\text{eff}}} Z(f) df - \Lambda W_{\text{eff}} = \Lambda W_{\text{eff}} - \int_{W_{\text{eff}}} \frac{N(f)}{|C(f)|^2} df \quad (15)
\]

Now, we can write for the threshold \(\Lambda\):

\[
\text{Figure 3 The frequency selective channel characteristics and a power spectral density } S_x(f) \text{ of } x(t)
\]
The PSD repartition in the channel is presented in the fig. 4. The hatched area is numerically equal with the power transmitted over the channel, \( P_x \) and the effective occupied bandwidth is \( W_e = [f_1, f_2] \cup [f_3, f_4] \). If the threshold \( \Lambda \) is greater than the absolute maximum of \( Z(f) \) over bandwith \( W \), then \( W_e = W \), else \( W_e < W \). The result is known as the water-filling theorem and shows that the power must be distributed to those zones of the channel having a small noise reflected at the entry.

Substituting (14) in (8) and by limiting the domain of the integral to \( W_e \), the effective occupied bandwidth for which \( S_x(f) > 0 \), we obtain the maximum values of capacity:

\[
C_{\text{max}} = \int_{W_e} \log_2 \left[ \frac{P_x}{N_0 W_e} + \frac{1}{W_e} \int_{w_e} df \frac{C(f)^2}{|C(f)|^2} \right] df
\]

(17)

For the band where \( S_x(f) = 0 \), the integral is zero. Substituting (3) in (17), we find that:

\[
C_{\text{max}} = \int_{W_e} \log_2 \left( \frac{\Lambda}{N(f)} \right) df
\]

(18)

### 2.2 Capacity, with optimal power distribution, for the channels affected by white noise

If the noise \( n(t) \) is white, then \( N(f) = N_0 = \text{const.} \) and

\[
Z(f) = \frac{N_0}{|C(f)|^2}
\]

(19)

Substituting \( N(f) = N_0 \) in the threshold \( \Lambda \) expression (16) we obtain:

\[
\Lambda = \frac{P_x}{W_e} + \frac{1}{W_e} \int_{w_e} df \frac{C(f)^2}{|C(f)|^2}
\]

(20)

Substituting (20) in (18) and making \( N(f) = N_0 \), we obtain the maximum capacity of the frequency selective channel with white noise at the output:

\[
C_{\text{max}} = \int_{W_e} \log_2 \left[ \frac{P_x}{N_0 W_e} + \frac{1}{W_e} \int_{w_e} df \frac{|C(f)|^2}{|C(f)|^2} \right] df
\]

\[
= \int_{W_e} \log_2 \left[ \frac{P_x}{N_0 W_e} + \frac{1}{W_e} \int_{w_e} df \frac{C(f)^2}{|C(f)|^2} \right] df - \int_{W_e} \log_2 \left( \frac{1}{|C(f)|^2} \right) df
\]

(21)

The first integrand being a constant, the integral is immediately. The relation (21) can be rewritten:

\[
C_{\text{max}} = W_e \log_2 \left[ \frac{P_x}{N_0 W_e} + \frac{1}{W_e} \int_{w_e} df \frac{C(f)^2}{|C(f)|^2} \right] - \frac{1}{W_e} \int_{W_e} \log_2 \left( \frac{1}{|C(f)|^2} \right) df
\]

(22)

This expression can be transformed (see annexe A.2):

\[
C_{\text{max}} = W_e \log_2 \left( \frac{\text{SNR}_e + |C(f)|^2}{|C(f)|^2} \right)
\]

(23)

where \( \text{SNR}_e \) is the signal to noise ratio calculated for the effective occupied bandwidth:

\[
\text{SNR}_e = \frac{W_e \log_2 (\text{SNR} + |C(f)|^2)}{|C(f)|^2}
\]

(24)
\[ SNR_e = \frac{P_x}{N_0 W_e} \]  \hspace{1cm} (24)

where

\[ |C(f)|^2 = \frac{1}{W_e} \int_{W_e} df |C(f)|^2 \]  \hspace{1cm} (25)

and

\[ |C(f)|^2 = \exp \left( \frac{1}{W_e} \int \ln |C(f)|^2 \right) df \]  \hspace{1cm} (26)

It comes out that \( f(x) \) is a mean value of \( f(x) \) for an interval and \( f(x) \) is a generalized geometrically mean that can be applied to the function for that interval.

We consider now a frequency nonselective channel, with frequency response \( K \) in the effective occupied bandwidth, which satisfies the equivalence condition:

\[ \int K^2 df = \int |C(f)|^2 df \]  \hspace{1cm} (27)

or

\[ K^2 = \frac{1}{W_e} \int |C(f)|^2 df = |C(f)|^2 \]  \hspace{1cm} (28)

Normalizing the channel, by division of \( |C(f)| \) by \( K \), we will have a channel for which:

\[ \langle C(f) \rangle^2 = \frac{1}{W_e} \int |C(f)|^2 df = 1 \]  \hspace{1cm} (29)

where we don’t use a different notation for the normalized value.

2.3 Capacity of the flat channel
We consider now a nonselective frequency channel, for which \( |C(f)| = K \), and the effective occupied bandwidth is even \( W \), so \( W_e = W \). We have also, according to (25) and (26):

\[ |C(f)|^2 = K^{-2} = K^{-2} \]  \hspace{1cm} (30)

\[ |C(f)|^2 = \exp \left( \frac{1}{W} \int \ln K^{-2} df \right) = K^{-2} \]  \hspace{1cm} (31)

and SNR, calculated over entire bandwidth, becomes SNR:

\[ \text{SNR} = \frac{S_s}{WN_0} \]  \hspace{1cm} (32)

Introducing (30)-(32) in (23) we will find \( C_p \), the nonselective flat channel capacity:

\[ C_p = W \log_2 \left( 1 + \frac{P_x K^2}{N_0 W} \right) = W \log_2 \left( 1 + \frac{P_y}{N_0 W} \right) \]  \hspace{1cm} (33)

where \( P_y \) is the received signal power and \( P_y/(N_0 W) \) is the signal to noise ratio at the receiver. In the case of normalized channel \( K=1 \), \( P_y = P_x \) and

\[ C_p = W \log_2 \left( 1 + \text{SNR} \right) \]  \hspace{1cm} (34)

2.4 Capacity loss of a frequency selective channel comparative to capacity of the flat channel
We will consider a normalized flat channel, with \( |C(f)| \) that respect the condition (29), and the equivalent flat channel having \( K=1 \). Both channels will be exploit at the SNR great enough so that \( W_e \approx W \). Then it can be calculated the loss of capacity caused by channel selectivity:

\[ \Delta C = C_{p,\text{flat}} - C_{\text{max}} \approx W \log_2 (1 + \text{SNR}) - K C(f) \right)^2 \]  \hspace{1cm} (35)

And it can be shown (see A.3 annex) that we have:

\[ \Delta C < -\frac{1}{\ln 2} \int \ln |C(f)|^2 df > 0 \]  \hspace{1cm} (36)

2.5 Channels capacity with optimal power distribution
We can renounce to maximize the capacity of the selective channel and consequently to distribute uniformly the power on the entire bandwidth. Then

\[ S_s = \frac{P_s}{W} \]  \hspace{1cm} (37)

So, psd of received signal, complying with (2) is:

\[ S_s(f) = \frac{P_s}{W} |C(f)|^2 + N_0 = \frac{N_0 W + P_s |C(f)|^2}{W} \]  \hspace{1cm} (38)

where \( N_0 \) is the white noise psd. The capacity of selective channel with uniform power distribution is \( C_u \):

\[ C_u = \int W \log_2 \left( 1 + \frac{P_s |C(f)|^2}{N_0 W} \right) df = \int W \log_2 \left[ 1 + \text{SNR} |C(f)|^2 \right] df \]  \hspace{1cm} (39)

where \( P_s \) is the received signal power.