A New Implementation of the Hyperanalytic Wavelet Transform

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Abstract—**The property of shift-invariance associated with a good directional selectivity are important for the application of a wavelet transform in many fields of image processing. Unfortunately, the 2D discrete wavelet transform is shift-variant and has a reduced directional selectivity. These disadvantages can be attenuated if a complex wavelet transform is used. In this paper, we propose a new implementation of such a wavelet transform, recently introduced, the hyperanalytic wavelet transform. It is quasi shift-invariant, it has a good directional selectivity, and a reduced degree of redundancy of 4 in 2D. The implementation proposed is very simple. The properties already mentioned are proved by simulation.**

I. Introduction

Over twenty year ago, Grossman and Morlet [1] developed the Continuous Wavelet Transform (CWT) [2], using continuous complex-valued mother wavelets. Initial analysis based on wavelet decompositions was implemented using such mother wavelets. Both magnitude and phase descriptions of non-stationary signals were determined, and an early example of analysis includes wavelet ridge methods proposed by Delprat *et al.* [3]. However subsequently for many years interest focused on the Discrete Wavelet Transform (DWT) and signal estimation. The DWT was developed to implement the wavelet transform of time-compact mother wavelets and as compact discrete wavelet filters cannot be exactly analytic [4], real wavelets were used. A revival of interest in later years has occurred in both signal processing and statistics for the usage of complex wavelets, [5], and in particular complex analytic wavelets [6]–[9]. This revival of interest may be linked to the development of complex-valued discrete wavelet filters [10] and the clever dual filter bank [6]. The complex wavelet transform has been shown to provide a powerful tool in signal and image analysis [11], where most of the properties of the transform follow from the analyticity of the wavelet function. In [12] were derived large classes of wavelets generalizing the concept of a 1-D local complex-valued analytic decomposition to a 2-D vector-valued hyperanalytic decomposition. In the present paper we propose a very simple implementation of this hyperanalytic wavelet transform, (HWT). We present in section II the case of 1D signals, highlighting the quasi shift-invariance of the implementation proposed. We compare the degree of shift invariance obtained using the proposed implementation with results obtained using other quasi-shift-invariant wavelet transforms, in terms of redundancy. The case of 2D signals is analyzed in section III, and the enhanced directional selectivity of the proposed implementation is proved. Section IV is dedicated to some concluding remarks.

II. The 1D signals case

A 1D wavelet transform (WT), is shift-sensitive if an input signal shift causes an unpredictable change of the transform coefficients. The shift-sensitivity of the DWT is generated by the down-samplers used for its computation. In [2, 13] is devised the un-decimated DWT (UDWT), which is a WT without down-samplers. Although the UDWT is shift-insensitive, it has high redundancy, 2^J (where J represents the number of iterations of the WT). Another disadvantage of the UDWT comes from the fact that it requires the implementation of a large number of different filters. In [14] was proposed a new shift-invariant but very redundant WT, named Shift Invariant Discrete Wavelet Transform, SIDWT. This proposition is based on a translation invariant algorithm. The algorithm introduced in [14] and called Cycle Spinning (CS) was conceived to suppress the artifacts in the neighborhood of discontinuities introduced by the DWT, and it implies the rejection of the translation dependency. For a range of delays, data is shifted, its DWT is computed, and than the result is un-shifted. Doing this for a range of shifts, and averaging the several results so obtained, a quasi shift-invariant DWT is implemented. The degree of redundancy of this transform is proportional to the number of shifts of the input signal produced. CS over the range of all circular shifts of the input signal is equivalent to SIDWT. In [15], is demonstrated that approximate shiftability is possible for the DWT with a small, fixed amount of transform redundancy. In this reference is designed a pair of real mother wavelets such that one is approximately the Hilbert transform of the other. This wavelet pair defines a complex discrete wavelet transform (CDWT) presented in figure 1 a). A complex wavelet coefficient is obtained by interpreting the wavelet coefficient from one DWT tree as being its real part, whereas the corresponding coefficient from the other tree is interpreted as its imaginary part. In [7] is developed the dual tree complex wavelet transform (DTCWT), which is a quadrature pair of DWT trees, similar to the CDWT. The DTCWT coefficients may be interpreted as arising from the DWT associated with a quasi-analytic wavelet. Both DTCWT and CDWT are invertible and quasi shift-invariant; however the design of these quadrature wavelet pairs is quite complicated and it can be done only through approximations.

The implementation of the HWT representing the aim of our proposal is presented in figure 1 b). We first apply a Hilbert transform to the data. The real wavelet transform is then applied to the analytical signal associated to the input data, obtaining complex coefficients. The two implementations of the CDWT presented in figure 1 are equivalent because:

$$\begin{aligned} d_{DTCWT}[m,n] &= \langle x(t), \psi_{m,n}(t) + i\mathcal{H}\{\psi_{m,n}(t)\}\rangle = \\ &= \langle x(t), \psi_{m,n}(t)\rangle - i\langle x(t), \mathcal{H}\{\psi_{m,n}(t)\}\rangle = \\ &= \langle x(t), \psi_{m,n}(t)\rangle + i\langle \mathcal{H}\{x(t)\}, \psi_{m,n}(t)\rangle = \\ &= \langle x(t) + i\mathcal{H}\{x(t)\}, \psi_{m,n}(t)\rangle = d_{HWT}[m,n] \end{aligned} \tag{1}$$

In fact neither the DTCWT nor the proposed implementation of HWT correspond to perfect analytic mother wavelets, because the exact digital implementation of a Hilbert transform pair of mother wavelets with good performance is not possible in the case of the first transform and because the digital Hilbert transformer is not a realizable system in the case of the second transform. The DTCWT requires special mother wavelets (the implementation of the HWT proposed in figure 1 b) can be realized using classical mother wavelets like those conceived by Daubechies) but can assure a higher degree of shift invariance. These two transforms have in the 1D case a redundancy of 2. In [16] is proposed a two-stage mapping-based complex wavelet transform (MBCWT) that consists of a mapping onto a complex function space followed by a DWT of the complex mapping computation. The authors of this article have observed that the DTCWT coefficients admit also another interpretation: they may be interpreted as the coefficients of a DWT applied to a complex signal associated with the input signal. The complex signal is defined as the Hardy-space image of the input signal. As the Hardy-space mapping of a discrete signal is impossible to compute, they have defined a new function space called the Softy-space, which is an approximation to Hardy-space. The HWT implementation

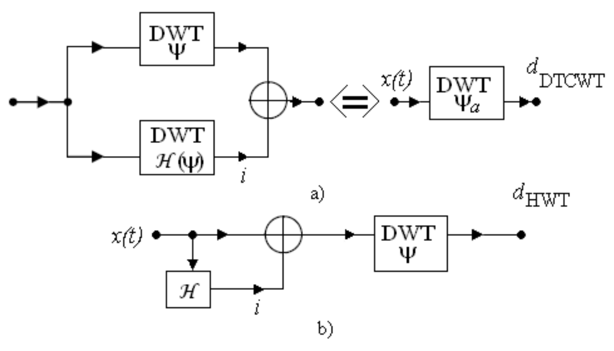

Figure 1. The implementations of the DTCWT a) and of the HWT b) are equivalent.

presented in figure 1 b) can be regarded like a MBCWT where the mapping system is a Hilbert transformer. The advantages of the MBCWT are:

- controllable redundancy of the mapping stage (there are non-redundant implementations of this transform);
- the possibility to use any mother wavelets, which provides flexibility to this transform (the proposed implementation of the HWT has also this advantage).

In order to evaluate the shift-invariance performance of our transform, we performed two types of simulations. First we made a comparison with the DWT where the shift-invariance is evaluated visually. In the first experiment, we used as input signal a unitary step, like in [7]. Second, we introduced a shift-invariance degree. In [17], is defined a new measure of the shift-invariance, called "shiftability". According to this definition, a transform is shiftable if and only if any subband energy of the transform is invariant under input-signal shifts. Although weaker than shift invariance, shiftability is important for applications because it is equivalent to interpolability, which is a property ensuring the preservation of transform subband energy under input-signal shifts. We introduced a new criterion: the degree of shift invariance. In order to calculate this measure, we calculate the energies of every set of detail coefficients (at different decomposition levels) and of the approximation coefficients, corresponding to a certain delay (shift) of the input signal samples. This way, we obtain a sequence of energies at each decomposition level, each sample of this sequence corresponding to a different shift. Then the mean m and the standard deviation d of every energy sequence are computed. Our degree of invariance is defined as:

$$\text{Deg}=1-d/m \tag{2}$$

In the second set of simulations we have compared the degree of shift-invariance of our transform with the degree of shift invariance of the CS with a various number of cycle spins and for a variety of spinning steps (a spinning step is the number of samples the signal is shifted once). In both experiments in order to isolate the coefficients corresponding to each level, after the computation of the corresponding forward WT, we put all the coefficients corresponding to the other levels to zero, by applying a "mask" on the sequence obtained. For a better understanding of this procedure, we illustrate in figure 2 the system used for the analysis of the shift invariance at the 3rd

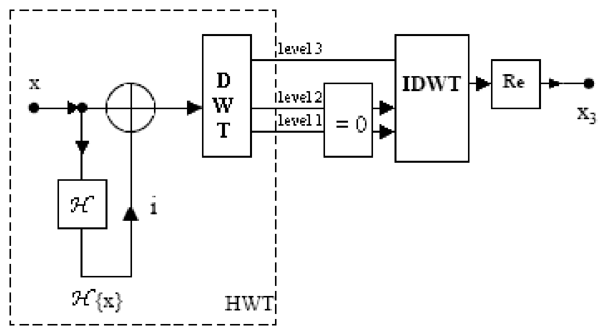

Figure 2. The system used for the shift-invariance analysis of the third level of the wavelet decomposition. In this example is considered the case of the proposed implementation of HWT.

decomposition level of the HWT. We repeated the simulations corresponding to the second experiment for several mother wavelets commonly used in the literature (Daubechies, Symmlet and Coiflet). The first experiment already described is illustrated in figure 3. It can be observed that the DWT is not shift-invariant; the lines of coefficients corresponding to different shifts are not parallel. The new implementation of HWT is quasi-shift-invariant. That is, for shifted version of the same signal applied to the input, shifted-like versions of the signal reconstructed are obtained following the steps indicated in figure 2. In fig. 4 is shown the dependency of the degree of shift invariance of the new implementation with respect to the regularity of the mother wavelets used for its computation. The Daubechies family was investigated, each element being indexed by its number of vanishing moments. As the curve illustrated in figure 4 indicates, the degree of shift-invariance increases with the regularity of the mother wavelets used. In table 1 is presented a comparison between the proposed implementation of the HWT and the CS. This comparison is based on the values of the degree of shift invariance calculated for the approximation coefficients obtained after the 3rd iteration of the corresponding WT computation algorithm (Scaling fn., level 3), for the detail coefficients obtained after the 3rd iteration (Wavelets level 3), for the detail coefficients obtained after the 2nd iteration (Wavelets level 2) and for the detail coefficients obtained after the 1st iteration (Wavelets level 1). It can be observed, analyzing this table, that our implementation of the HWT is equivalent to the CS with redundancy 64, from the degree of shift-invariance point of view.

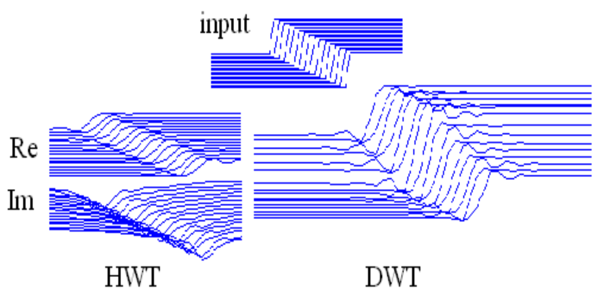

Figure 3. A comparison between the HWT and the DWT at the first decomposition level for approximation coefficients.

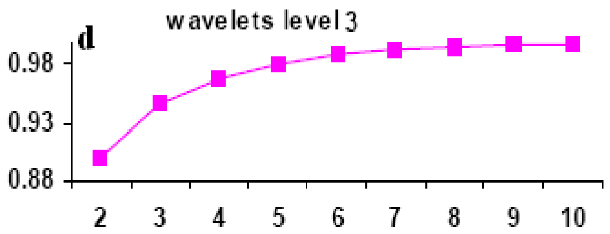

Figure 4. The dependency of the degree of shift-invariance of HWT on the regularity of the mother wavelet used for its computation.

TABLE I. A COMPARISON BETWEEN TWO QUASI SHIFT-INVARIANT WAVELET TRANSFORMS THE HWT AND THE CS.

Symmlet, 10	**HWT**	**CS , st.=1, 64 del.**	**CS, st.=1, 512 del.**
Redundancy	2	64	512
Scal. fn. Level 3	0.8594	0.7551	0.7551
Wavelets level 3	0.9962	0.9962	0.9995
Wavelets level 2	0.9963	0.9965	0.9996
Wavelets level 1	0.9992	0.9985	0.9998
Daubechies, 10	**HWT**	**CS, st.=1, 64 del.**	**CS, st.=1, 512 del.**
Scal. fn. Level 3	0.8594	0.7551	0.7551
Wavelets level 3	0.9981	0.9965	0.9996
Wavelets level 2	0.9982	0.9968	0.9996
Wavelets level 1	0.9992	0.9985	0.9998

This in an excellent result, given that the HWT has a redundancy of 2 only, since for L samples to its input we get L complex samples in the wavelet domain.

III. THE 2D SIGNALS CASE

All the WTs already mentioned have simpler or more complicated 2D generalizations. The generalization of the analyticity concept in 2D is not obvious, because there are multiple definitions of the Hilbert transform in this case. In the following we will use the definition of the analytic signal associated to a 2D real signal named hypercomplex signal. So, the hypercomplex mother wavelets associated to the real mother wavelets $\psi(x,y)$ is defined as:

$$\begin{aligned}\psi_a(x,y) &= \psi(x,y) + i\mathcal{H}_x\{\psi(x,y)\} + \\ &+ j\mathcal{H}_y\{\psi(x,y)\} + k\mathcal{H}_x\{\mathcal{H}_y\{\psi(x,y)\}\}\end{aligned} \tag{3}$$

where $i^2 = j^2 = -k^2 = -1,\ and\ ij = ji = k$, [18].

The HWT of the image $f(x,y)$ is:

$$HWT\{f(x,y)\} = \langle f(x,y), \psi_a(x,y)\rangle.$$

Tacking into account relation (3) it can be written:

$$\begin{aligned} &HWT\{f(x,y)\} = DWT\{f(x,y)\} + \\ &iDWT\{\mathcal{H}_x\{f(x,y)\}\} + jDWT\{\mathcal{H}_y\{f(x,y)\}\} + \\ &+kDWT\{\mathcal{H}_y\{\mathcal{H}_x\{f(x,y)\}\}\} = \\ &\langle f_a(x,y), \psi(x,y)\rangle = DWT\{f_a(x,y)\}. \end{aligned} \tag{4}$$

So, the 2D-HWT of the image $f(x,y)$ can be computed with the aid of the 2D-DWT of its associated hypercomplex image. In consequence the HWT implementation proposed in this paper uses four trees, each one implementing a 2D-DWT. The first tree is applied to the input image. The second and the third trees are applied to 1D Hilbert transforms computed across the lines ($\mathcal{H}_x$) or columns ($\mathcal{H}_y$) of the input image. The fourth tree is applied to the result obtained after the computation of the two 1D Hilbert transforms of the input image. The enhancement of the directional selectivity of the 2D-HWT is realized like in the case of the 2D-DTCWT, [7,11], by the separation of the real and imaginary parts of the complex coefficients belonging to each subband of the WT.

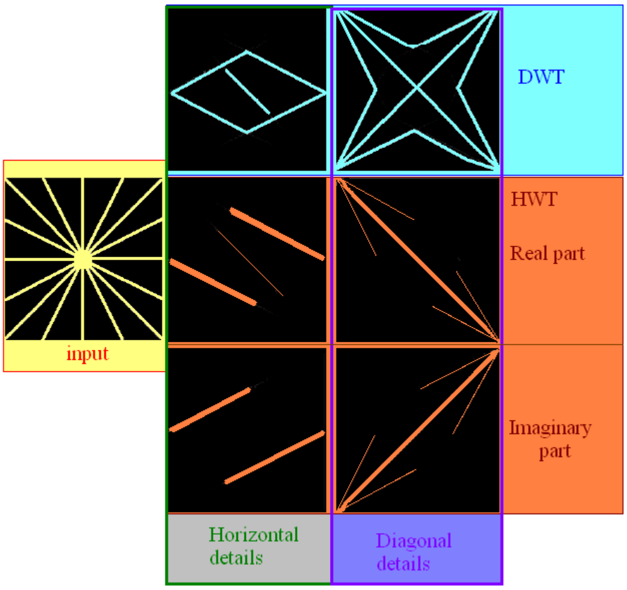

Figure 5. The absolute values of the spectra of horizontal and diagonal detail sub-images obtained after the first iterations of DWT and HWT (proposed implementation). In the HWT case, the real and imaginary parts of complex coefficients are separated.

A comparison of the directional selectivity of the DWT and HWT, implemented as proposed in this paper, is presented in figure 5. Like the 2D-DTCWT, the 2D-HWT, implemented as proposed in this paper, has six preferential orientations: $\pm\text{atan}(1/2)$, $\pm\pi/4$ and $\pm\text{atan}(2)$. The 2D-DWT has only three preferential orientations: 0, $\pi/4$ and $\pi/2$, it do not make the difference between the two principal diagonals. The better directional selectivity of the proposed implementation of 2D-HWT versus the 2D-DWT can be easily observed, comparing the corresponding detail sub-images in figure 5. For the diagonal detail sub-images, for example, the imaginary part of the HWT rejects the directions: $\text{-atan}(1/2)$, $-\pi/4$ and $\text{-atan}(2)$, whereas the DWT conserves these directions.

IV. CONCLUSION

The HWT is a very modern WT as it has been formalized only a year ago [12]. In this paper we have proposed a very simple implementation of the HWT, which does not require the construction of any special wavelet filters and permits the exploitation of the mathematical results and of the algorithms previously established in the evolution of wavelets theory. It has a very flexible structure, as we can use any orthogonal or biorthogonal real mother wavelets and it is fast. As one can notice from the simulation results presented in the previous chapters, a good selection of the mother wavelet confers a high degree of shift invariance. The proposed implementation leads to both a higher degree of shift-invariance than the CS transform at the same redundancy and to an enhanced directional selectivity in the 2D case, when the same mother wavelets is used. In the theoretical derivations reported in this paper we have considered an ideal Hilbert transformer. Our research team will make a closer analysis on the effects of using realisable Hilbert transformers as the next step. Further comparisons of the proposed implementation of the HWT with the DTCWT will permit us to conclude how the disadvantages of this idea, highlighted in [11], restrict the application of the proposed implementation. We intend to use it for denoising, segmentation and watermarking of real object images, in the future.

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