ADAPTIVE SAMPLING RATE OBTAINED USING WAVELETS

Tibor Asztalos, Dorina Isar, Alexandru Isar

Electronics and Telecommunications Faculty
Technical University Timișoara,
2 Bd. V. Parvan, 1900
Romania, isar@ee.utt.ro

ABSTRACT
For storage purposes, an adaptive sampling rate is recommended for the digitisation of a signal with unknown bandwidth. This procedure is equivalent with the sampling of the continuous in time signal followed by an adaptive compression of the discrete in time signal obtained. The aim of this paper is an adaptive data compression method based on wavelet theory. This method suppose the adaptive selection of the wavelet's mother used, for the maximisation of the compression factor. The implementation of the correspondent algorithm and a working example for the proposed method are presented.

1. HOW TO OBTAIN AN ADAPTIVE SAMPLING RATE?
One of the most important parameter for a data acquisition system is the sampling frequency. The selection of the value of this parameter is difficult especially when the bandwidth of the signal that must be processed is unknown (a good example is a digital signal recorder). This is the reason why usually is preferred to use the higher sampling frequency that can be technologically obtained. Doing so, the data stream obtained after the sampling of a narrow bandwidth low-pass signal is very redundant. For storage purposes, an adaptive sampling rate is recommended for the digitisation of a signal with unknown bandwidth. The simplest way to built a sampling system with varying sampling rate is to use the higher sampling frequency for the digitisation of the continuous in time signal, followed by a new sampling of the discrete in time signal obtained. The second sampling uses an adaptive sampling rate. The sampling frequency (for this second sampling) can be selected in accordance with the instantaneous bandwidth of the discrete in time signal obtained after the first sampling [1]. The procedure already described is equivalent with the sampling of the continuous in time signal followed by an adaptive compression of the discrete in time signal obtained. There are a lot of compression methods [2]. One of them is based on an orthogonal transform followed by a digitisation [3]. One of the orthogonal transforms that can be used is the Discrete Wavelet Transform (DWT). This is a versatile transform that can be computed very fast. The system proposed in this paper is presented in figure 1. This system is composed by a continuous in time sampling system (CTSS) (containing an analog to digital converter too) and a discrete in time system. A block for the computation of the DWT of signal x[n], DWT {x[n]}, and an adaptive tresholding system (ATS) composes this second system. These systems realise the adaptive sampling. The last block in figure realises the reconstruction of the signal x[n]. The output signal u[n] represents the result of the adaptive sampling of the signal x[n]. The output signal y[n] represents the reconstruction, from its samples, of the signal x[n]. Due to the block ATS the mean square reconstruction error is kept under an imposed level, representing a fixed percent of the energy of the signal x[n]. So, the signal y[n] can be viewed like the sampled version of the signal x(t) and the signal u[n] can be used for storage purposes because the number of non-zero samples of this signal is very small.

Figure 1. An adaptive sampling system.

This adaptive compression method is presented in section 2. The matching to the input signal is realised by selecting the wavelet's mother used for the computation of the DWT. This selection process is described in section 3. In section 4 is presented an example for the adaptive compression method proposed. Finally the last section is dedicated to the conclusion of this paper.

2. ADAPTIVE COMPRESSION USING WAVELETS
Ingrid Daubechies [4] and Stephan Mallat [5] introduced the DWT. This is an orthogonal transform with two parameters:

- The type of wavelet's mother used,
- The number of iterations, M.

It's use in data compression is recommended because the signal DWT[x[n]] has more small samples or zero samples than the signal x[n]. Neglecting these small samples, a compressed version of the signal x[n] is obtained. This manifestation of the DWT is due to its properties:
P1. (The whitening property) When N increases to ∞ the DWT computation system acts like a whitening filter [6].

P1'. The convergence speed of DWT{x[n]}, when x[n] is a stationary random signal, to a white noise, increases when the number of vanishing moments of the wavelet's mother used increases [7].

P2. The wavelet transforms (DWT and IDWT) conserve the energy, [8]:

\[
E_x = \sum_{n=-\infty}^{\infty} x[n]^2 = \sum_{n=-\infty}^{\infty} (DWT\{x[n]\})^2 = \sum_{n=-\infty}^{\infty} (IDWT\{x[n]\})^2
\]

The property P1 specifies the number of iterations of the wavelet transforms recommended for data compression applications. This number must be the greatest possible. The property P1' recommends the use of a wavelet's mother with the greatest number of vanishing moments possible.

The property P2 indicates a possibility for the adaptive control of the approximation mean square error of the signal that must be compressed with the signal reconstructed from the compressed version. The mean square error for the approximation of x[n] with the signal y[n] is equal with the mean square error induced by the adaptive thresholding system. The input-output relation for this system is:

\[
u[n] = \begin{cases} DWT\{x[n]\}, & \text{if } |DWT\{x[n]\}| > T \\ 0, & \text{if not.} \end{cases}
\]

The approximation mean square error induced by this system, for finite duration input signals, has the following value:

\[
e = E\left(\sum_{n} (DWT\{x[n]\} - u[n])^2\right) = \sum_{k=1}^{K} (DWT\{x[n_k]\})^2 \quad (1)
\]

where \(n_k\) represents the indexes of samples of the signal \(DWT\{x[n]\}\) smaller than the threshold T. Let this error be a percent, \(\alpha\), of the energy of the signal \(x[n]\), \(E_x\):

\[
e = \alpha E_x \quad (2)
\]

The threshold T must be selected for an imposed value of \(\alpha\) such that (1) to be verified. This relation is an equation in K. Let \(z[n]\) be the sequence obtained taking the samples of the sequence \(DWT\{x[n]\}\) in the inverse order of their magnitude. The mean square approximation error of \(DWT\{x[n]\}\) by \(u[n]\) (in (1)) is equal with:

\[
e = \sum_{k=1}^{K} z[k]^2
\]

So, the value of the threshold T can be selected to be equal with \(z[K+1]\). We have the algorithm for the selection of the threshold presented in figure 2. Hence the value of the threshold can be automatically selected after the generation of the sequence \(z[n]\).

The value of the compression ratio, that can be obtained, for a specified signal, depends on the type of the wavelet's mother selected for the computation of the DWT and IDWT in figure 1. In the following we present an adaptive method for the selection of the wavelet's mother that maximises the compression factor of a specified signal.

![Figure 2. A fast adaptive threshold selection algorithm.](image)

3. WAVELETS AND POLYNOMIALS

Every signal \(x(t)\) can be approximated by a polynomial \(P_l(t)\) of order \(P_l\) (it's Taylor decomposition) in an interval \(I_l\) with a specified error. So, the input signal for the system in figure 1 can be written as:

\[
x(t) = \sum_{l} P_l(t), t \in I_l \quad (4)
\]

where \(I_l\) are consecutive disjoints intervals. So the signal \(x(t)\) can be segmented, every segment representing a polynomial. The degree of every approximation polynomial can be determined imposing a superior bound of the approximation error of the original signal with that polynomial. The support of the approximation polynomial can be determined in the same manner. But a polynomial of order \(P_l\) (the approximation of the signal \(x(t)\) in the lth segment) can be exactly decomposed in a space \(V_{\alpha l}\), generator of a multiresolution analysis \(\{V_{\alpha l}\}_{\alpha \in Z}\) [4]. This Hilbert space is generated by a scaling function \(\phi_0(t)\), that corresponds to a wavelet's mother \(\psi_{\alpha l}(t)\) that has a number of vanishing moments equal with \(P_l + 1\) [9]. This observation gives us the possibility to formulate one of most important results of this paper:

P3. The best compression of the l segment of the signal \(x(t)\) can be obtained using a wavelet's mother with \(P_l + 1\) vanishing moments.
Proof. Let us compute the details of the polynomial $P_{p_l}(t)$ of order $p_l$ in the interval $I_1$:

$$\rho^p_d[m]=\langle P_{p_l}(t), \psi_{p_l}(t-m) \rangle =$$

$$=\left\langle P_{p_l}(t), 2^\frac{p}{2} \psi_{0}(2^p t - m) \right\rangle$$

where $p$ represents the order of the iteration in the DWT. But our polynomial can be expressed in the form:

$$P_{p_l}(t)=\sum_{k=0}^{p_l} a_k t^k$$

for $t \in I_1$

So, the details of this signal at the scale $p$ are:

$$\rho^p_d[m]=\sum_{k=0}^{p_l} a_k \left\langle t^k, 2^\frac{p}{2} \psi_{0}(2^p t - m) \right\rangle$$

where $t$ is in the interval $[m_0, M_0]$. But:

$$\left\langle t^k, 2^\frac{p}{2} \psi_{0}(2^p t - m) \right\rangle =$$

$$=2^\frac{p}{2} \int_{m_0}^{M_0} t^k \psi_{0}(2^p t - m) dt$$

Using the new variable:

$$2^p t - m = v$$

we can write:

$$\left\langle t^k, 2^\frac{p}{2} \psi_{0}(2^p t - m) \right\rangle =$$

$$=2^\frac{p}{2} \int_{m_0}^{M_0} v^k \psi_{0}(v) dv$$

or:

$$\left\langle t^k, 2^\frac{p}{2} \psi_{0}(2^p t - m) \right\rangle =$$

$$=\sum_{m=0}^{m_l-1} C^p_k \left[ \int_{m_i-m}^{2^p M_i-m} v^k \psi_{0}(v) dv \right] m^{k-o}$$

where $\psi_{0}(v)$ is a Coiflet scaling function of order 1.

Case I. The support of the wavelet $\psi_{0}(t)$ is included in the interval:

$$I_{pml} = [2^p m_{l-1}-m, 2^p M_{l-1}-m]$$

In this case we can write:

$$\int_{2^p m_{l-1}-m}^{2^p M_{l-1}-m} v^o \psi_{0}(v) dv = \int_{-\infty}^{\infty} v^o \psi_{0}(v) dv = 0$$

because:

$$o \leq k \leq p_l$$

and the wavelet's mother $\psi_{0}(t)$ has $p_{l+1}$ vanishing moments. So, in this case all the details of the polynomial considered are nulls:

$$\rho^p_d = 0$$

Case II. The support of the wavelet $\psi_{0}(t)$ is not included in the interval $I_{pml}$ but their intersection is not empty. In this case the details are not nulls:

$$\rho^p_d \neq 0$$

Case III. The intersection of the support of the wavelet's mother $\psi_{0}(t)$ with the interval $I_{pml}$ is empty. In this case the details are nulls, too. Hence only in the second case the details are not nulls. If we want to use for the computation of the DWT a wavelet's mother with a smaller number of vanishing moments then some details in case I are not nulls. So, the entire number of details not nulls is greater than the correspondent number in the case already studied. If we want to use for the computation of the DWT a wavelet's mother with a greater number of vanishing moments then all the details in cases III and I rest nulls. But this wavelet's mother has a longer support than the wavelet's mother $\psi_{0}(t)$. This is the reason why there are more details in the case II. So, in this situation too, the entire number of details not nulls is greater then the correspondent number in the case already studied. Hence the smallest number of details not nulls is obtained when the DWT of the signal considered is computed using the wavelet's mother $\psi_{0}(t)$. So, the property P3 is proved. Like a consequence of the property P3 we can formulate a new sampling theorem. When a Coiflet wavelet's mother is used then the number of vanishing moments of the functions $\phi_{0}(t)$ is 1. Using such a scaling function we can formulate a new sampling theorem dedicated to polynomial signals.

TL. Any polynomial $P_{p_l}(t)$, of degree $l$ can be perfectly reconstructed from its samples $P_{p_l}(k)$, using the reconstruction relation:

$$P_{p_l}(t) = \sum_{k=0}^{\infty} P_{p_l}(k) \phi_{0}(t - k)$$

where $\phi_{0}(t)$ is a Coiflet scaling function of order 1.

The proof is based on the fact that $P_{p_l}(t)$ is a member of the space $V_0$ generated by the Coiflet scaling function of order 1. Is very simple to prove that:

$$\left\langle P_{p_l}(t), \phi_{0}(t - k) \right\rangle = P_{p_l}(k)$$

Finally, we can prove that the proposed compression method is robust. In the following we present an example of adaptive sampling of a signal using the adaptive compression method already described.

4. AN EXAMPLE

A program in C for the simulation of the acquisition system in figure 1 was realised. The signal $x(t)$ is approximated with a polynomial of degree $P_{l+1}$, between 0 and $P_{max}$, obtained by Lagrange interpolation, using equally spaced points, on the interval $I_1$. The support $I_1$ is obtained using the following algorithm. First is considered the support of the entire signal $x(t)$. The signal is approximated with polynomials with...
increasing degrees. For every degree is computed the approximation error of the signal \( x(t) \), realised by the current polynomial, in every point. If there is a point where the approximation error is superior to an imposed level, then the degree of the approximation polynomial is increased. If the degree of the interpolation polynomial becomes greater than \( P_{\text{max}} \) and a point where the approximation error is superior to the imposed level still exists then the support is divided in two intervals with equal length and the approximation by interpolation process is restarted for each interval. This partition of the support is stopped in two situations:
1. In every new interval is founded a good polynomial approximation (the approximation error is smaller than the imposed level) with a degree smaller or equal with \( P_{\text{max}} \);
2. The length of the new interval becomes smaller than an imposed value. Then for the length of the interval is selected this imposed value and for the degree of the polynomial is selected the value \( P_{\text{max}} \).

For the signal in figure 3, using a value of 0.5% for \( \alpha \) we have obtained a value of 7.65 for the compression factor.

![Figure 3](image3.png)

**Figure 3.** The signal that must be sampled, \( x(t) \).

The segmentation of the signal in figure 3 is presented in table 1.

<table>
<thead>
<tr>
<th>The order number of the segment</th>
<th>The degree of the corresponding polynomial</th>
<th>The duration of the segment [number of samples]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>128</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>128</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>128</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>256</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
<td>64</td>
</tr>
</tbody>
</table>

**Table 1.** The polynomial interpolation of the signal in figure 3.

The reconstructed signal \( y[n] \) is presented in figure 4.

![Figure 4](image4.png)

**Figure 4.** The signal reconstructed after the adaptive sampling of the signal \( x(t) \), \( y[n] \).

5. CONCLUSION

An equivalent sample density can be computed using the value of the compression ratio obtained. The equivalent sample density represents the ratio between the number of samples not nulls in the signal \( u[n] \) and the number of samples of the signal \( x[n] \). Using this equivalent sample rate an equivalent sampling frequency \( f_{\text{se}} \) can be computed. This equivalent sampling frequency represents the product of the initial sampling frequency (that used to sample the continuous in time signal \( x(t) \)) and the corresponding equivalent sample rate.

The value \( f_{\text{se}} \) represents the adaptive sampling rate mentioned in the title of this paper. The compression method proposed in this paper realises a greater compression factor at the same distortion level that the compression methods reported in [1], [10], [11] and [12].

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6. REFERENCES