Adaptive Capturing Transient Signals Using Wavelets

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Abstract
When capturing transient signals, the maximum recording time of the measurement is limited by the high sample rate used to preserve input signal details. To obtain a maximum recording time keeping the details an adaptive compression must be performed. Such a method is proposed in this paper. The method is based on the use of the discrete wavelet transform. To optimise the compression ratio a criteria for the matching of the wavelets mother with the input signal is proposed. This matching problem is very important in the wavelet's literature. The correspondent wavelets mother searching method is described. Some simulations results are described. The superiority of the proposed compression method is proved.

1. Data Adaptive Compression
When capturing transient signals with a digital oscilloscope or a waveform recorder, it is desirable to use a high sample rate to preserve input signal details. This limits the maximum recording time of the measurement. To obtain a longer recording time the data obtained by sampling must be compressed. When there is significant detail present into the input signal the compression ratio must be small and this ratio can be high when there is no significant detail present. Thus the compression device must maximises the recording time without compromising signal integrity. So, an adaptive compression method must be used.

There are many modern methods for data compression, [1]. These methods can be classified into two categories:
- loss less compression,
- compression with loss.
This second class is usually preferred because the magnitude of the compression factor obtained is more important. The methods in this second class use the decorrelation effect of an orthogonal transform. The Karhunen-Loève transform is optimal for this purpose, when the model of the signal that must be processed is a gaussian process. In practice, the Discrete Cosine Transform (DCT) is used [2]. This is not an optimal transform but can be computed faster. It converges to the Karhunen - Loève transform. Another orthogonal transform used in data compression is the discrete wavelet transform (DWT). This transform converges to the Karhunen-Loève transform, too [3], [4]. When the model of the signal that must be processed is a non-linear random process the discrete wavelet transform is more useful than the Karhunen-Loève transform [5]. The problem for the compression with loss is the trade-off between the magnitude of the compression factor and the amount of distortion (the measure of loss) in the signal obtained after the reconstruction phase (the recovered signal).

The aim of this paper is a method for the selection of the wavelet's mother that maximise the compression factor in accordance with the signal that must be processed for an imposed reconstruction distortion level.
In [6] another data adaptive compression method based on the use of the Short Time Fourier Transform is proposed. In the following we shall prove the superiority of the method proposed in this paper.
2. Data Compression with Wavelets

The compression system proposed in this paper is presented in the following figure:

![Diagram of the compression system](image)

**Fig. 1: The compression system**

The threshold detector (TD) performs the operation:

\[
\begin{align*}
    z[n] = &\begin{cases} 
    y[n] \text{ if } |y[n]| > \lambda \\
    0 \text{ if } |y[n]| \leq \lambda 
    \end{cases}
\end{align*}
\]

where \( \lambda \) is the magnitude of the threshold. The step of the adaptive quantifier (Q) is of magnitude \( 2\lambda \). The coder (C) performs a supplementary lossless compression. The output of the compression system is described by the signal \( v[n] \). The decoder (D) is the inverse system of the coder. At the output of the system that computes the inverse wavelet transform (IDWT), the recovered signal \( \hat{x}[n] \) is obtained. The distortion caused by the compression has the value:

\[
D = E\left\{ \|x[n] - \hat{x}[n]\|^2 \right\}
\]

where \( E \) represents the random average operator.

Because the DWT and IDWT are orthogonal transforms, the last relation becomes:

\[
D = E\left\{ \|y[n] - u[n]\|^2 \right\}
\]

The magnitude of the threshold \( \lambda \) is selected to satisfy the condition:

\[
D \leq \alpha E_x, \quad \alpha < 1
\]

where \( E_x \) represents the energy of the input signal, \( x[n] \). We can prove the following proposition:

**P1. A superior bound of the distortion of the recovered signal \( \hat{x}[n] \) obtained after the adaptive compression with wavelets, is \( N\epsilon^2 \), where \( N \) represents the number of input signal samples.**

**Proof.** The mean square approximation error of the signal \( y[n] \) by the signal \( z[n] \) is:

\[
\epsilon_1 = E\left\{ \|y[n] - z[n]\|^2 \right\} = \sum_{k=1}^{K} (y[n_k] - y[n_k])^2
\]

where \( n_k \) represents the positions of samples of the signal \( y[n] \) with the absolute value inferior to the threshold value \( \lambda \). There are \( K \) such samples. Let \( o[n] \) be the signal obtained after the samples of \( y[n] \) were sorted in order of their magnitude. The mean square error becomes:

\[
\epsilon_1 = \sum_{k=1}^{K} (o[n_k])^2 \leq K\lambda^2
\]

if the quantisation step used is equal with \( \lambda \), then the mean square quantisation error is:

\[
\epsilon_2 = \sum_{n=1}^{N} (z[n] - u[n])^2
\]
For every sample of the sequence $o[k]$, $k=1,K$, a zero in the sequence $u[n]$ is associated. For the other samples of the signal $z[n]$ the difference $z[n]-u[n]$ is inferior to the value $\lambda$. This is the reason why it can be written:

$$\varepsilon_2 \leq \sum_{k=K+1}^{N} \lambda^2 = (N-K)\lambda^2$$

So, the proposition is proved.

Therefore, to keep the distortion under the value $\alpha E_x$ is sufficient to select the threshold with the value: $\lambda = \sqrt{\frac{\alpha E_x}{N}}$. The compression factor is proportional with the number of samples rejected by the threshold detector. So, we must maximise this number keeping the value of $D$ under an imposed value.

The parameters of the DWT are:
- the wavelet’s mother $\psi(t)$,
- the number of iterations, $M$.

Because the DWT converges asymptotically to the Karhunen-Loève transform for $M \to \infty$, [3], the better is to utilise the greatest value possible for $M$ that can be obtained.

The aim of this paper is to choose in accordance with the input signal the better wavelet’s mother.

For the same input signal and the same threshold value, different compression factors can be obtained, when different wavelet’s mothers are selected for the computation of the discrete wavelet transform. Hence an extra maximisation of the compression factor, for the same distortion level in the recovered signal, can be obtained. This can be done by an optimal selection of the wavelet's mother used to compute the DWT and the IDWT, based on the parameters of the input signal. Using this optimal wavelet's mother, a discrete wavelet transform with the maximum number of null coefficients is obtained.

3. Selecting the Wavelet's Mother

The matching between the input signal and the wavelet's mother is one of the most interesting problems in wavelet theory. Let the signal $x(t)$ be a polynomial of degree $P$:

$$x(t) = \sum_{k=0}^{P} a_k t^k$$

and $\psi_r(t)$ a wavelet’s mother with $r$ vanishing moments:

$$\int_{-\infty}^{\infty} t^k \psi_r(t) dt = 0 ; k = 0, r-1$$

It's easy to prove the following proposition:

P2. If $P<r$ then all the coefficients $d_m[n]$ are nulls.

Proof. Let us compute the details of the discrete wavelet transform of the signal $x(t)$:

$$d_m[n] = \langle P_p(t), \psi_{m}^p(t-m) \rangle = \left\langle \frac{P_p(t), 2^m \psi_{m}^p(2^m t-m) \right\rangle$$

where $p$ represents the order of the iteration in the DWT.

So, the details of this signal at the scale $p$ are:
\[ P_d[m] = \sum_{k=0}^{P} a_k \left( t^k \cdot 2^P \psi_{0l}(2^P t - m) \right) \]

But:
\[ \left( t^k \cdot 2^P \psi_{0l}(2^P t - m) \right) = 2^P \int_{-\infty}^{\infty} t^k \psi_{0l}(2^P t - m) dt \]

Using the new variable: \( 2^P t - m = v \)
we can write:
\[ \left( t^k \cdot 2^P \psi_{0l}(2^P t - m) \right) = 2^{kP} \int_{-\infty}^{\infty} (v + m)^k \psi_{0l}(v) dv \]

But:
\[ \int_{-\infty}^{\infty} v^k \psi_{0l}(v) dv = 0, \quad 0 \leq k \leq P \]

because the wavelet's mother \( \psi_{0l}(t) \) has \( P+1 \) vanishing moments. So, in this case all the details of the polynomial considered are nulls.

So, for the compression of the polynomial \( x(t) \) the better wavelet's mother is one with a number of vanishing moments greater than the degree of the polynomial. Doing so, the maximum number of null coefficients, after the computation of the corresponding discrete wavelet transform of the considered signal, is obtained. This is the reason why the compression factor can be maximised using this method. The problem is to estimate the degree of the input polynomial.

**In the following, we propose a method that performs the segmentation of the signal that must be processed, in polynomial segments.** Every segment of this signal can be processed with a discrete wavelet transform, based on a wavelet's mother with a number of vanishing moments superior with 1 to the degree of the polynomial on the corresponding segment. The input signal, \( x(t) \), can be developed, at every moment \( t \), into a Taylor series:
\[ x(t) = \sum_{k=0}^{P} \frac{P}{k!} x^{(k)}(t_0)(t - t_0)^k + R_{P+1}(t) \]

The first term in the right hand side is a polynomial function \( P_P(t) \) of degree \( P \). This is a good approximation for the signal \( x(t) \), only for \( t \) contained into an interval, \( I_0 \), that contains the moment \( t_0 \). So:
\[ x(t) = P_P(t) + R_P(t) \]

The magnitude of \( P \) can be selected such that \( P_P(t) \) to be a good approximation of \( x(t) \) in \( I_0 \). We can obtain the following polynomial approximation of the signal \( x(t) \):
\[ x(t) = \begin{cases} P_{P_0}(t), & t \in I_0 \\ P_{P_1}(t), & t \in I_1 \\ \vdots \end{cases} \]

Every polynomial function \( P_{P_k}(t) \) is member of a space \( V_{0k} \) that generates a multiresolution analysis of \( L^2(R) \), [7]. This multiresolution analysis is generated by a scaling function that corresponds to a wavelet's mother with \( k+1 \) vanishing moments, [8]. So to process with wavelets the polynomial function \( P_{P_k}(t) \) we can use the wavelet's mother DAU \( l+1 \) with \( l > k \). Hence, every segment of the signal \( x(t) \) (defined by the interval \( I_k \)) must be processed with the aid of a known wavelet's mother.
4. A Segmentation Algorithm

For a function described by polynomials with different degrees on different segments we have proved the following proposition:

P3. For every polynomial segment, \( P_p(t) \), the greatest compression factor is obtained when a wavelet’s mother with \( P_{p+1} \) vanishing moments is used.

Proof. Let us compute the details of the polynomial \( P_p(t) \) of order \( P \) in the interval \( I \) :

\[ P_p(t) = \sum_{k=0}^{P} a_k t^k \quad \text{for} \quad t \in I \]

where \( p \) represents the order of the iteration in the DWT. But our polynomial can be expressed in the form:

\[ P_p(t) = \sum_{k=0}^{P} a_k t^k \quad \text{for} \quad t \in I \]

So, the details of this signal at the scale \( p \) are:

\[ p \cdot d[m] = \sum_{k=0}^{P} a_k \langle t^k, 2^p \psi_0 \rangle \]

where \( t \) is in the interval \( I = [m, M] \). But:

\[ \langle t^k, 2^p \psi_0 \rangle = 2^p \int_{m}^{M} t^k \psi_0 \quad dt \]

Using the new variable: \( 2^p t - m = v \)

we obtain:

\[ \langle t^k, 2^p \psi_0 \rangle = 2^p \int_{m}^{M} (v + m)^k \psi_0 \quad dv \]

or:

\[ \langle t^k, 2^p \psi_0 \rangle = \sum_{o=0}^{P} C_k^o \int_{2^p m - m}^{2^p M - m} v^o \psi_0 \quad dv \quad m^{k-o} \]

We have three distinct situations:

Case I. The support of the wavelet \( \psi_0(t) \) is included in the interval:

\[ I_{pm} = [2^p m - m, 2^p M - m] \]

In this case we can write:

\[ \int_{2^p m - m}^{2^p M - m} v^o \psi_0 \quad dv = \int_{-\infty}^{\infty} v^o \psi_0 \quad dv = 0 \]

because:

\[ o \leq k \leq P \]

and the wavelet’s mother \( \psi_0(t) \) has \( P+1 \) vanishing moments. So, in this case all the details of the polynomial considered are nulls:

\[ p \cdot d = 0, (\forall) p \]
Case II. The support of the wavelet $\psi_0(t)$ is not included in the interval $I_{pml}$, but their intersection is not empty. In this case some details are not nulls:

$$\forall d \neq 0$$

Case III. The intersection of the support of the wavelet's mother $\psi_0(t)$ with the interval $I_{pml}$ is empty. In this case the details are nulls, too.
Hence only in the second case some details are not nulls.

If we want to use for the computation of the DWT a wavelet's mother with a smaller number of vanishing moments then some details in case I are not null too. So, the entire number of details not nulls is greater than the correspondent number in the case already studied.
If we want to use for the computation of the DWT a wavelet's mother with a greater number of vanishing moments then all the details in cases III and I rest nulls. But this wavelet's mother has a longer support than the wavelet's mother $\psi_0(t)$. This is the reason why there are more details non nulls in the case II. So, in this situation too, the entire number of details not nulls is greater than the correspondent number in the case already studied. Hence the smallest number of details not nulls is obtained when the DWT of the signal considered is computed using the wavelet's mother $\psi_0(t)$. So, the property P3 is proved.

The problem is to estimate the degree of the approximation polynomial on each segment and to identify the segment. This can be done starting from the initial signal $x(t)$, supposed with compact support. This support represents the first segment. A polynomial interpolation cycle for this signal can be performed using some values of the signal taken in integer points. These points must be quasi equally spaced in the whole segment. When a number of $K$ such points is used then the interpolation polynomial has the degree $K-1$. $K$ must be increased between 1 and a maximal value corresponding to the maximum number of the vanishing moments of the wavelet's mother that is intended to be used, $K_{max}$. For every value of $K$ the approximation error of the signal $x(t)$ is computed. If there are some values of this error, superior to an imposed level of error, then $K$ must be increased. If not, the current value of $K$ is retained. This represents the polynomial degree searched. If $K$ becomes greater than $K_{max}$ then the support of $x(t)$ must be divided in two quasi equally segments and the procedure already presented is repeated. If an admissible value of $K$ is obtained on one of those segments then this represents the polynomial degree searched on the corresponding segment. If there is one segment where the value of $K$ becomes greater than $K_{max}$ then this segment must be divided in two new quasi equally spaced segments. This procedure must be stopped when the segments become too short. If on such a segment $K$ becomes greater than $K_{max}$ then the value $K_{max}$ will be used.
Based on this algorithm, a program in C was written. The compression simulations based on this program are very promising.

5. An Example

For the signal in figure 2, using a value of 0.5% for $\alpha$ we have obtained a value of 7.65 for the compression factor.

The segmentation of the signal in figure 3 is presented in table 1.

<table>
<thead>
<tr>
<th>The order number of the segment</th>
<th>The degree of the corresponding polynomial</th>
<th>The duration of the segment [number of samples]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>128</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>128</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>128</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>256</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
<td>64</td>
</tr>
</tbody>
</table>

Table I. The polynomial interpolation of the signal in figure 3.
Fig. 2: The signal that must be sampled, x[n].

Fig. 3: The signal reconstructed after the adaptive sampling of the signal x[n], y[n].

6. Conclusion

This paper describes one of the steps of a compression method based on the use of the discrete wavelet transform. This method supposes the computation of the discrete wavelet transform, an adaptive threshold detection, an adaptive quantification and a loss less compression based on the use of a coder. In this paper we focus on the discrete wavelet transform. The maximisation of the compression factor, at an imposed distortion level of the reconstructed signal, can be obtained if the wavelet mother, used to compute the discrete wavelet transform, is adaptively selected to match with the signal that must be processed. The matching is realised by selecting a wavelet’s mother with a number of vanishing moments specified. We have presented a segmentation method of the input signal that gives different segments of this signal. Each segment contains a signal that can be approximated with a polynomial, with a degree that is estimated. The discrete wavelet transform of this segment must be computed using a wavelet mother with a number of vanishing moments superior with 1 to the degree of the polynomial. We have compared the method proposed in [6] with the method proposed in this paper using the same input signals and we have obtained a greater compression factor when our method was used. Other useful references are [9]…[13].

References


