A Second Order Statistical Analysis of the 2D Discrete Wavelet Transform

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Abstract—**We present a general second order statistical analysis of the 2D Discrete Wavelet Transform (DWT) resulted after the computation of the correlation functions in all possible cases: inter-scale and inter-band dependency, inter-scale and intra-band dependency and intra-scale and intra-band dependency. The expected value and the variance of the wavelet coefficients are also computed. The resulting equations are useful for the design of different signal processing systems based on the wavelet theory.**

Keywords- DWT, statistics, intercorrelation, autocorrelation, mean, variance, inter-scale dependency, intra-scale dependency.

I. INTRODUCTION

A great number of Wavelet Transforms (WT) can be used to represent images. The first one was the 2D Discrete Wavelet Transform, 2D DWT, [1]. All the WTs are characterized by two parameters: the mother wavelets, MW and the primary resolution, PR (number of iterations). The importance of their selection is highlighted in [2]. An appealing particularity of the 2D DWT is the interscale dependency of the wavelet coefficients.

II. 2D DWT IMPLEMENTATION

The main advantage of the proposed implementation of the 2D DWT is its flexibility, as it inherits some of the classes of mother wavelets developed in the framework of the 1D DWT, like the Daubechies, Symmlet or Coiflet families. This is why this implementation is adequate to a multi-wavelet environment. Each of the iterations of the algorithm used for the computation of the 2D DWT implies several operations. First, the lines of the input image (obtained at the end of the previous iteration) are passed trough two different filters (a lowpass filter having the impulse response m_0 and a high-pass filter m_1) resulting two different sub-images. Then the lines of the two sub-images obtained at the output of the two filters are decimated with a factor of 2. Next, the columns of the two images obtained are low-pass filtered with m_0 and high-pass filtered with m_1. The columns of those four sub-images are also decimated with a factor of 2. Four new sub-images, representing the result of the current iteration (which corresponds to the current decomposition level (or scale)), are obtained. These sub-images are called subbands. The first sub-image, obtained after two lowpass filtering, is named approximation sub-image (or LL subband). The other three are named detail sub-images: LH, HL and HH. The LL sub-image represents the input for the next iteration. In the following, the coefficients of the DWT will be denoted with ${}_xD_m^k$, where x represents the image who's DWT is computed (considered as a bivariate random signal), m represents the current scale and k = 1 - for the subband LH, k = 2 - for HL, k = 3 - for HH and k = 4 - for LL. These coefficients are computed using the following relation:

$${}_xD_m^k[n_1,p_1] = \left\langle x(\tau_1,\tau_2), \psi_{m,n_1,p_1}^k(\tau_1,\tau_2)\right\rangle, \quad (6)$$

where the wavelets are real functions and can be factorized as:

$$\psi_{m,n,p}^k(\tau_1,\tau_2) = \alpha_{m,n}^k(\tau_1)\cdot\beta_{m,p}^k(\tau_2), \quad (7)$$

and the two factors can be computed using the scale function $\varphi(\tau)$ and the mother wavelets $\psi(\tau)$ with the aid of the following relations:

$$\alpha_{m,n}^k(\tau) = \begin{cases}\varphi_{m,n}(\tau), & k=1,\ 4\\ \psi_{m,n}(\tau), & k=2,\ 3\end{cases} \quad (8)$$

$$\beta_{m,p}^k(\tau) = \begin{cases}\varphi_{m,p}(\tau), & k=2,\ 4\\ \psi_{m,p}(\tau), & k=1,\ 3\end{cases}, \quad (9)$$

where:

$$\varphi_{m,n}(\tau) = 2^{-\frac{m}{2}}\varphi\left(2^{-m}\tau-n\right) \text{ and}$$

$$\psi_{m,n}(\tau) = 2^{-\frac{m}{2}}\psi\left(2^{-m}\tau-n\right). \quad (10)$$

Taking into account equations (8)-(10) it can be written:

$$\psi_{m,n,p}^k(\tau_1,\tau_2) = 2^{-m}\psi^k\left(2^{-m}\tau_1-n, 2^{-m}\tau_2-p\right)$$
$$\text{where } \psi^k(\tau_1,\tau_2) = \psi^k{}_{0,0,0}(\tau_1,\tau_2). \quad (11)$$

III. SECOND ORDER STATISTICAL ANALYSIS

We begin the statistical analysis by computing the mean of the wavelet coefficients:

$$\mu_{{}_xD_m^k} = E\left\{{}_xD_m^k\right\} = E\left\{\left\langle x(\tau_1,\tau_2), \psi_{m,n,p}^k(\tau_1,\tau_2)\right\rangle\right\} =$$

$$= E\left\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau_1,\tau_2)\cdot \psi_{m_1,n_1,p_1}^{k_1*}(\tau_1,\tau_2)\, d\tau_1 d\tau_2 \right\}, \quad (12)$$

Applying Fubini's theorem we obtain:

$$\mu_{{}_xD_m^k} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\{x(\tau_1,\tau_2)\}\psi_{m_1,n_1,p_1}^{k_1*}(\tau_1,\tau_2) d\tau_1 d\tau_2 = \\ = \mu_x \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_{m_1,n_1,p_1}^{k_1*}(\tau_1,\tau_2) d\tau_1 d\tau_2 = \\ = \mu_x \mathcal{F}\left\{\psi_{m_1,n_1,p_1}^{k_1}(\tau_1,\tau_2)\right\}(0,0). \quad (13)$$

Because the spectrum of the wavelet can be expressed as,

$$\mathcal{F}\left\{\psi_{m,n,p}^{k}\right\}(\xi_1,\xi_2) = 2^m \cdot e^{-j2^m(\xi_1 n+\xi_2 p)} \cdot \\ \cdot \mathcal{F}\left\{\psi^k\right\}\left(2^m\xi_1, 2^m\xi_2\right), \quad (14)$$

the last equation becomes:

$$\mu_{{}_xD_m^k} = \begin{cases} 0, & k = 1,2,3 \\ 2^m \cdot \mu_x, & k = 4. \end{cases} \quad (15)$$

Consequently, the expected values of the detail wavelet coefficients are null. Only the means of the sub-images in the fourth sub-band (corresponding to the approximation coefficients) are not null but dependent of the scale and of the mean of the original image μ_x.

The intercorrelation of two wavelet coefficients belonging to the subbands k_1 and k_2 and to the scales m_1 and $m_2 = m_1 + q$ and having the geometrical coordinates (n_1, p_1) and (n_2, p_2) respectively can be computed using the following relation:

$$R_{{}_xD_{m_1}^{k_1}\,{}_xD_{m_2}^{k_2}}(n_1-n_2, p_1-p_2) = E\left\{ {}_xD_{m_1}^{k_1}\left({}_xD_{m_2}^{k_2}\right)^* \right\} = \\ = E\Big\{\left\langle x(\tau_1,\tau_2), \psi_{m_1,n_1,p_1}^{k_1}(\tau_1,\tau_2)\right\rangle \cdot \\ \cdot \left\langle x(\tau_1,\tau_2), \psi_{m_2,n_2,p_2}^{k_2}(\tau_1,\tau_2)\right\rangle^* \Big\} = \\ = E\Big\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau_1,\tau_2)\psi_{m_1,n_1,p_1}^{k_1*}(\tau_1,\tau_2) d\tau_1 d\tau_2 \cdot \\ \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^*(\tau_3,\tau_4)\psi_{m_2,n_2,p_2}^{k_2}(\tau_3,\tau_4) d\tau_3 d\tau_4 \Big\}. \quad (16)$$

Taking into account the linearity of the expectation, the last equation becomes:

$$R_{{}_xD_{m_1}^{k_1}\,{}_xD_{m_2}^{k_2}}(n_1-n_2, p_1-p_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_x(\tau_1-\tau_3, \tau_2-\tau_4)\cdot \\ \cdot \psi_{m_1,n_1,p_1}^{k_1*}(\tau_1,\tau_2)\cdot \psi_{m_2,n_2,p_2}^{k_2}(\tau_3,\tau_4)\, d\tau_1 d\tau_2 d\tau_3 d\tau_4 = \\ = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(R_x * \psi_{m_2,n_2,p_2}^{k_2}\right)(\tau_1,\tau_2)\cdot \psi_{m_1,n_1,p_1}^{k_1*}(\tau_1,\tau_2)\, d\tau_1 d\tau_2. \quad (17)$$

Applying Parseval's identity, it can be written:

$$R_{{}_xD_{m_1}^{k_1}\,{}_xD_{m_2}^{k_2}}(n_1-n_2, p_1-p_2) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_x(\xi_1,\xi_2)\cdot \\ \cdot \mathcal{F}\left\{\psi_{m_2,n_2,p_2}^{k_2}\right\}(\xi_1,\xi_2)\cdot \mathcal{F}^*\left\{\psi_{m_1,n_1,p_1}^{k_1}\right\}(\xi_1,\xi_2)\, d\xi_1 d\xi_2, \quad (18)$$

where $S_x(\xi_1,\xi_2)$ denotes the power spectral density of the input signal. So, for $m_2 = m_1 + q$, applying (14), equation (18) becomes:

$$R_{{}_xD_{m_1}^{k_1}\,{}_xD_{m_2}^{k_2}}(n_1-n_2, p_1-p_2) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_x(\xi_1,\xi_2)\cdot \\ \cdot 2^{2m_1}\cdot 2^q \cdot e^{-j\cdot 2^{m_1}\left(\xi_1\left(2^q n_2 - n_1\right)+\xi_2\left(2^q p_2 - p_1\right)\right)} \cdot \\ \cdot \mathcal{F}\left\{\psi^{k_2}\right\}\left(2^{m_1}\cdot 2^q \xi_1, 2^{m_1}\cdot 2^q\xi_2\right)\cdot \\ \cdot \mathcal{F}^*\left\{\psi^{k_1}\right\}\left(2^{m_1}\xi_1, 2^{m_1}\xi_2\right) d\xi_1 d\xi_2. \quad (19)$$

Taking into account the presence of decimators into the 2D DWT computation algorithm, the product of the Fourier transforms from the right hand side of the last equation does not make sense for all the values of ξ_1 and ξ_2. In fact, it can be written:

$$\mathcal{F}\left\{\psi^{k_2}\right\}\left(2^{m_1}\cdot 2^q \xi_1, 2^{m_1}\cdot 2^q\xi_2\right)\cdot \\ \cdot \mathcal{F}^*\left\{\psi^{k_1}\right\}\left(2^{m_1}\xi_1, 2^{m_1}\xi_2\right) = \\ \mathcal{F}\left\{\psi^{k_2}\right\}\left(2^{m_1}\cdot 2^q \xi_1, 2^{m_1}\cdot 2^q\xi_2\right)\cdot \\ \cdot \mathcal{F}^*\left\{\psi^{k_1}\right\}\left(2^{m_1}\cdot 2^q\xi_1, 2^{m_1}\cdot 2^q\xi_2\right), \quad (20)$$

and the equation (19) becomes:

$$R_{{}_xD_{m_1}^{k_1}\,{}_xD_{m_2}^{k_2}}(n_1-n_2, p_1-p_2) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_x(\xi_1,\xi_2)\cdot \\ \cdot 2^{2m_1}\cdot 2^q \cdot e^{-j\cdot 2^{m_1}\left(\xi_1\left(2^q n_2 - n_1\right)+\xi_2\left(2^q p_2 - p_1\right)\right)} \cdot \\ \cdot \mathcal{F}\left\{\psi^{k_2}\right\}\left(2^{m_1}\cdot 2^q \xi_1, 2^{m_1}\cdot 2^q\xi_2\right)\cdot \\ \cdot \mathcal{F}^*\left\{\psi^{k_1}\right\}\left(2^{m_1}\cdot 2^q\xi_1, 2^{m_1}\cdot 2^q\xi_2\right) d\xi_1 d\xi_2. \quad (21)$$

Making the changes of variables $\nu_1 = 2^{m_1}\cdot 2^q \cdot \xi_1$ and $\nu_2 = 2^{m_1}\cdot 2^q\cdot\xi_2$, the equation (21) becomes:

$$R_{{}_xD_{m_1}^{k_1}\,{}_xD_{m_2}^{k_2}}(n_1-n_2, p_1-p_2) = \\ = \frac{1}{4\pi^2}\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}\int_{(2l_1-1)\pi}^{(2l_1+1)\pi}\int_{(2l_2-1)\pi}^{(2l_2+1)\pi} S_x\left(2^{-m_1}2^{-q}\nu_1, 2^{-m_1}2^{-q}\nu_2\right)\cdot \\ \cdot 2^{-q}\cdot e^{-j\left(\nu_1\left(n_2-2^{-q}n_1\right)+\nu_2\left(p_2-2^{-q}p_1\right)\right)} \cdot \\ \cdot \mathcal{F}\left\{\psi^{k_2}\right\}(\nu_1,\nu_2)\cdot \mathcal{F}^*\left\{\psi^{k_1}\right\}(\nu_1,\nu_2)\, d\nu_1 d\nu_2, \quad (22)$$

or after new changes of variables, $\omega_1 = \nu_1 - 2l_1\pi$ and $\omega_2 = \nu_2 - 2l_2\pi$, it can be written:

$$R_{{}_xD_{m_1}^{k_1}\,{}_xD_{m_2}^{k_2}}\left(2^q n'_1 - n_2, 2^q p'_1 - p_2\right) =$$
$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} S_x\left(2^{-m_1}2^{-q}\left(\omega_1 + 2l_1\pi\right),\right.$$
$$\left.2^{-m_1}2^{-q}\left(\omega_2 + 2l_2\pi\right)\right)\cdot 2^{-q}\cdot e^{-j\left(\omega_1\left(n_2-n_1'\right)+\omega_2\left(p_2-p_1'\right)\right)}\cdot \tag{23}$$
$$\mathcal{F}\left\{\psi^{k_2}\right\}\left(\omega_1 + 2l_1\pi, \omega_2 + 2l_2\pi\right)\cdot$$
$$\mathcal{F}^*\left\{\psi^{k_1}\right\}\left(\omega_1 + 2l_1\pi, \omega_2 + 2l_2\pi\right)d\omega_1 d\omega_2.$$

We have taken once again into account the effect of decimators, putting $n_1 = 2^q n_1'$ and $p_1 = 2^q p_1'$. Using the notation:

$$G\left(\omega_1,\omega_2\right) = 2^{-q}\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} S_x\left(2^{-m_1}2^{-q}\left(\omega_1 + 2l_1\pi\right),\right.$$
$$\left.2^{-m_1}2^{-q}\left(\omega_2 + 2l_2\pi\right)\right)\cdot\mathcal{F}\left\{\psi^{k_2}\right\}\left(\omega_1 + 2l_1\pi, \omega_2 + 2l_2\pi\right)\cdot \tag{24}$$
$$\mathcal{F}^*\left\{\psi^{k_1}\right\}\left(\omega_1 + 2l_1\pi, \omega_2 + 2l_2\pi\right),$$

the equation (23) can be written in the final form:

$$R_{{}_xD_{m_1}^{k_1}\,{}_xD_{m_2}^{k_2}}\left(2^q n'_1 - n_2, 2^q p'_1 - p_2\right) =$$
$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} G\left(\omega_1,\omega_2\right) e^{-j\left(\omega_1\left(n_2-n_1'\right)+\omega_2\left(p_2-p_1'\right)\right)} d\omega_1 d\omega_2, \tag{25}$$

which represents a 2D inverse Fourier transform in discrete time. So, the inter-scale and inter-band dependency of the wavelet coefficients ${}_xD_{m_1}^{k_1}$ and ${}_xD_{m_2}^{k_2}$ has the expression:

$$R_{{}_xD_{m_1}^{k_1}\,{}_xD_{m_2}^{k_2}}\left(2^q n'_1 - n_2, 2^q p'_1 - p_2\right) = g\left[n_2 - n_1', p_2 - p_1'\right]. \tag{26}$$

The product of the last two factors in the right hand side of (23) represents the inter-spectrum of mother wavelets ψ^{k_1} and ψ^{k_2}:

$$S_{\psi^{k_2}\psi^{k_1}}\left(\omega_1,\omega_2\right) = \mathcal{F}\left\{\psi^{k_2}\right\}\left(\omega_1,\omega_2\right)\cdot\mathcal{F}^*\left\{\psi^{k_1}\right\}\left(\omega_1,\omega_2\right). \tag{27}$$

In consequence the function $G\left(\omega_1,\omega_2\right)$ can be expressed in the equivalent form:

$$G\left(\omega_1,\omega_2\right) = 2^{-q}\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} S_x\left(2^{-m_1}2^{-q}\left(\omega_1 + 2l_1\pi\right),\right. \tag{28}$$
$$\left.2^{-m_1}2^{-q}\left(\omega_2 + 2l_2\pi\right)\right)\cdot S_{\psi^{k_2}\psi^{k_1}}\left(\omega_1 + 2l_1\pi, \omega_2 + 2l_2\pi\right),$$

which represents the spectrum of a continuous in time image ideally sampled with a unitary step, $g\left(\tau_1,\tau_2\right)$. Taking into account the Wiener-Hincin theorem for the power spectral density S_x, we can find the analytical expression of this signal:

$$g\left(\tau_1,\tau_2\right) = 2^{2m_1+q} R_x\left(2^{m_1+q}\tau_1, 2^{m_1+q}\tau_2\right) * R_{\psi^{k_2}\psi^{k_1}}\left(\tau_1,\tau_2\right), \tag{29}$$

which substituted in (26) gives the final form of the inter-scale and inter-band intercorrelation of the wavelet coefficients:

$$R_{{}_xD_{m_1}^{k_1}\,{}_xD_{m_2}^{k_2}}\left(2^q n'_1 - n_2, 2^q p'_1 - p_2\right) =$$
$$= 2^{2m_1+q} R_x\left(2^{m_1+q}\left(n_2 - n_1'\right), 2^{m_1+q}\left(p_2 - p_1'\right)\right) * . \tag{30}$$
$$* R_{\psi^{k_2}\psi^{k_1}}\left(n_2 - n_1', p_2 - p_1'\right).$$

As one can observe, the **inter-scale and inter-band dependency** of the wavelet coefficients depends on the autocorrelation of the input signal, R_x and on the intercorrelation of the mother wavelets that generate the considered sub-bands, $R_{\psi^{k_2}\psi^{k_1}}$. A simplified version of the equation (30) is obtained if we suppose that the input signal is a bi-dimensional i.i.d. white Gaussian noise with variance σ_w^2 and mean zero, $x\left(\tau_1,\tau_2\right) = w\left(\tau_1,\tau_2\right)$:

$$R_{{}_wD_{m_1}^{k_1}\,{}_wD_{m_2}^{k_2}}\left(2^q n'_1 - n_2, 2^q p'_1 - p_2\right) =$$
$$= 2^{2m_1+q}\cdot\sigma_w^2\cdot\delta\left[n_2 - n_1', p_2 - p_1'\right] *$$
$$* R_{\psi^{k_2}\psi^{k_1}}\left(n_2 - n_1', p_2 - p_1'\right) = \tag{31}$$
$$= 2^{2m_1+q}\cdot\sigma_w^2\cdot R_{\psi^{k_2}\psi^{k_1}}\left(n_2 - n_1', p_2 - p_1'\right).$$

In this case, the inter-scale and inter-band dependency is function on the intercorrelation of the mother wavelets generating the considered sub-bands only. Generally speaking, *the 2D DWT correlates the input signal.*

For $k_1 = k_2 = k$, the intercorrelation of the wavelet coefficients expressed by (30) becomes an inter-scale and intra-band dependency:

$$R_{{}_xD_{m_1}^{k}\,{}_xD_{m_2}^{k}}\left(2^q n'_1 - n_2, 2^q p'_1 - p_2\right) =$$
$$= 2^{2m_1+q} R_x\left(2^{m_1+q}\left(n_2 - n_1'\right), 2^{m_1+q}\left(p_2 - p_1'\right)\right) * . \tag{32}$$
$$* R_{\psi^{k}}\left(n_2 - n_1', p_2 - p_1'\right).$$

If the mother wavelet ψ^k generates by translations and dilations an **orthogonal** basis of $L^2\left(R\right)$ then its autocorrelation has the following property:

$$R_{\psi^k}\left(n,p\right) = \delta\left[n,p\right], \tag{33}$$

and the expression of the **inter-scale and intra-band dependency** becomes:

$$R_{{}_xD_{m_1}^{k}\,{}_xD_{m_2}^{k}}\left(2^q n'_1 - n_2, 2^q p'_1 - p_2\right) =$$
$$= 2^{2m_1+q} R_x\left(2^{m_1+q}\left(n_2 - n_1'\right), 2^{m_1+q}\left(p_2 - p_1'\right)\right). \tag{34}$$

This intercorrelation is function of the autocorrelation of the input signal only. If the input is a bi-dimensional i.i.d. white Gaussian noise with variance σ_w^2 and mean zero, $x\left(\tau_1,\tau_2\right) = w\left(\tau_1,\tau_2\right)$:

$$R_{{}_xD_{m_1}^{k}\,{}_xD_{m_2}^{k}}\left(2^q n'_1 - n_2, 2^q p'_1 - p_2\right) =$$

$$= 2^{2m_1+q} \cdot \sigma_w^2 \cdot \delta\left(2^{m_1+q}\left(n_2 - n_1'\right), 2^{m_1+q}\left(p_2 - p_1'\right)\right). \quad (35)$$

We can conclude from the equation above that *the wavelet coefficients with different resolutions are not correlated inside a sub-band*.

For $m_1 = m_2 = m$ the intercorrelation of the wavelet coefficients expressed by (34) becomes an **intra-scale and intra-band dependency**:

$$\begin{aligned} &R_{{}_xD_m^k}\left(n'_1 - n_2, p'_1 - p_2\right) = \\ &= 2^{2m} R_x\left(2^m\left(n_2 - n_1'\right), 2^m\left(p_2 - p_1'\right)\right). \end{aligned} \quad (36)$$

This autocorrelation of the wavelet coefficients is function of only the autocorrelation of the input signal. The last equation can be put in the equivalent form:

$$\begin{aligned} &R_{{}_xD_m^k}\left(n_1 - n_2, p_1 - p_2\right) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_x\left(2^{-m}\omega_1, 2^{-m}\omega_2\right)\cdot \\ &\cdot e^{-j\left[\omega_1\left(n_2-n_1\right)+\omega_2\left(p_2-p_1\right)\right]} d\omega_1 d\omega_2. \end{aligned} \quad (37)$$

Taking in (37) the limit for $m \to \infty$, we obtain:

$$R_{{}_xD_\infty^k}\left(n_1 - n_2, p_1 - p_2\right) = S_x(0,0)\cdot\delta\left[n_2 - n_1, p_2 - p_1\right]. \quad (38)$$

We can affirm that, asymptotically the 2D DWT transforms every colored noise into a white one. Hence this transform can be regarded as a *whitening system in an intra-band and intra-scale scenario*.

A similar result can be obtained if the input is a bi-dimensional i.i.d. white Gaussian noise with variance σ_w^2 and mean zero, $x(\tau_1,\tau_2) = w(\tau_1,\tau_2)$. In this case relation (37) becomes:

$$R_{{}_wD_m^k}\left(n_1 - n_2, p_1 - p_2\right) = \sigma_w^2 \cdot \delta\left[n_2 - n_1, p_2 - p_1\right]. \quad (39)$$

We can state that, *in the same band and at the same scale, the 2D DWT does not correlate the i.i.d. white Gaussian noise*.

Finally, the variances of the wavelet coefficients can be computed. For *k*=1 or 2 or 3:

$$\begin{aligned} &\sigma^2_{{}_xD_m^k} = E\left\{\left|{}_xD_m^k\left[n_1,p_1\right]\right|^2\right\} = R_{{}_xD_m^k}(0,0) = \\ &\overset{(37)}{=} \frac{1}{4\pi^2}\int_{R^2} S_x\left(2^{-m}\omega_1, 2^{-m}\omega_2\right) d\omega_1 d\omega_2 = \\ &= \mathcal{F}^{-1}\left\{S_x\left(2^{-m}\omega_1, 2^{-m}\omega_2\right)\right\}(0,0). \end{aligned} \quad (40)$$

Taking in (40) the limit for $m \to \infty$, we obtain:

$$\sigma^2_{{}_xD_\infty^k} = S_x(0,0). \quad (41)$$

The same result is obtained considering $x(\tau_1,\tau_2) = w(\tau_1,\tau_2)$.

For *k*=4,

$$\begin{aligned} &\sigma^2_{{}_xD_m^4} = E\left\{\left|{}_xD_m^4\left[n_1,p_1\right] - \mu_{{}_xD_m^4}\right|^2\right\} = \\ &= R_{{}_xD_m^4}(0,0) - \mu^2_{{}_xD_m^4} = \\ &= \mathcal{F}^{-1}\left\{S_x\left(2^{-m}\omega_1, 2^{-m}\omega_2\right)\right\}(0,0) - 2^{2m}\cdot\mu_x^2. \end{aligned} \quad (42)$$

If the input is a bi-dimensional i.i.d. white Gaussian noise with variance σ_w^2 and mean zero, $x(\tau_1,\tau_2) = w(\tau_1,\tau_2)$, then the last equation becomes:

$$\sigma^2_{{}_wD_m^4} = \sigma_w^2 - 2^{2m}\cdot\mu_w^2 = \sigma_w^2. \quad (43)$$

IV. Conclusions

We have established formulas describing the inter-scale and inter-band; inter-scale and intra-band and intra-scale and intra-band dependencies, respectively, of the coefficients of the 2D DWT. To the best of our knowledge the expressions of the inter-scale and inter-band as well as inter-scale and intra-band dependencies are original. We have also computed the expected values and the variances of the wavelet coefficients belonging to the same band and having the same scale. These equations are useful for the design of different image processing systems which apply the 2D DWT for compression, denoising, watermarking, segmentation or classification and to develop a second order statistical analysis of some complex 2D WTs. We have extensively treated the case when the input image is a bi-dimensional i.i.d. white Gaussian noise with variance σ_w^2 and null expectation, which is very useful for the design of denoising systems for the reduction of the additive white Gaussian noise with the aid of the 2D DWT. We have also reported some asymptotic results which prove that the 2D DWT can be regarded as a sub-optimal bi-dimensional whitening system.

Our objective for the near future is to extend the results obtained in this paper for the second order statistical analysis of the Hyperanalytic Wavelet Transform, HWT [3] and to apply it for image denoising in association with maximum a posteriori (MAP) filters [4]. To accomplish this task, a model for the distribution of the wavelet coefficients must be also found. So another future direction for our team will be the research of such a model.

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References

[1] D. L. Donoho, I. M. Johnstone, "Ideal spatial adaptation by wavelet shrinkage", Biometrika, 81(3) : 425-455, 1994.

[2] G. P. Nason, "Choice of wavelet smoothness, primary resolution and threshold in wavelet shrinkage", Statistics and Computing, 12, 2002, 219-227.

[3] I. Adam, C. Nafornita, J-M Boucher and A. Isar, "A New Implementation of the Hyperanalytic Wavelet Transform", Proc. IEEE Sympo. ISSCS 2007, Iasi, Romania, 401-404.

[4] I. Firoiu., C. Nafornita, J.–M. Boucher, A. Isar, "Image Denoising Using a New Implementation of the Hyperanalytic Wavelet Transform", IEEE Transactions on Instrumentation and Measurements, vol. 58, Issue 8, August 2009, pp. 2410-2416.