A Statistical Analysis of the 2D Discrete Wavelet Transform

Alexandru Isar[1], Sorin Moga[2], and Xavier Lurton[3]

[1] University POLITEHNICA
2 Bd. V. Parvan,
1900 Timisoara, Romania
(e-mail:isar@etc.utt.ro)
[2] GET - ENST Bretagne - CNRS UMR 2872
Technopôle de Brest Iroise, CS 83818,
29238 BREST Cedex, France
(e-mail: sorin.moga@enst-bretagne.fr)
[3] IFREMER - Centre de Brest
Service Acoustique et Sismique (TMSI/AS)
BP 70 Plouzan 29280 France
(e-mail: lurton@ifremer.fr)

Abstract. The aim of this paper is a complete statistical analysis of the two dimensional discrete wavelet transform, 2D DWT. The probability density function and the correlation of the coefficients of this transform are computed. The asymptotic behaviour of this transform is also studied. The results obtained were used to design a new denoising system dedicated to the processing of SONAR images.
Keywords: Discrete Wavelet Transform, Asymptotic analysis, convergence speed.

1 Introduction

The 2D DWT is a very modern mathematical tool. It is used in compression (JPEG 2000) and denoising applications. To exploit all its advantages, it must be carefully analysed. The aim of this paper is the study of this transform from the statistical point of view. Such a complete study was not already reported.

2 The 2D DWT

At every iteration of the DWT the lines of the input image (obtained at the end of the previous iteration) are low-pass filtered with a filter having the impulse response m_0 and high-pass filtered with the filter m_1. Then the lines of the two images obtained at the output of the two filters are decimated with a factor of 2. Next, the columns of the two images obtained are low-pass filtered with m_0 and high-pass filtered with m_1. The columns of those four images are also decimated with a factor of 2. Four new images (representing the result of the current iteration) are obtained. The first one, obtained after two low-pass filterings is named approximation image (or LL image). The others three are named detail images: LH, HL and HH. The LL image represents the input for the next iteration. In the following, the coefficients of the DWT will be noted with ${}_xD_m^k$, where x represents the image who's DWT is computed, m represents the iteration index and $k = 1$, for the HH image, $k = 2$, for the HL image, $k = 3$, for the LH image and $k = 4$, for the LL image. These coefficients are computed using the following relation:

$$ {}_xD_m^k\left[n_1, p_1\right] = \left\langle x\left(\tau_1, \tau_2\right), \psi_{m,n,p}^k\left(\tau_1, \tau_2\right)\right\rangle \tag{1} $$

where the wavelets can be factorized:

$$ \psi_{m,n,p}^k\left(\tau_1, \tau_2\right) = \alpha_{m,n,p}^k\left(\tau_1\right) \cdot \beta_{m,n,p}^k\left(\tau_2\right) \tag{2} $$

and the two factors can be computed using the scale function $\varphi\left(\tau\right)$ and the mother wavelets $\psi\left(\tau\right)$ with the aid of the following relations:

$$ \alpha_{m,n,p}^k\left(\tau\right) = \begin{cases} \varphi_{m,n}\left(\tau\right), \ k = 1, 4 \\ \psi_{m,n}\left(\tau\right), \ k = 2, 3 \end{cases} \tag{3} $$

$$ \beta_{m,n,p}^k\left(\tau\right) = \begin{cases} \varphi_{m,n}\left(\tau\right), \ k = 2, 4 \\ \psi_{m,n}\left(\tau\right), \ k = 1, 3 \end{cases} \tag{4} $$

where:

$$ \varphi_{m,n}\left(\tau\right) = 2^{-\frac{m}{2}}\varphi(2^{-m}\tau - n) \tag{5} $$

$$ \psi_{m,n}\left(\tau\right) = 2^{-\frac{m}{2}}\psi\left(2^{-m}\tau - n\right) \tag{6} $$

3 The pdf of the wavelet coefficients

The pdf of the wavelet coefficients, ${}_xD_m^k$, can be expressed with the aid of the pdf of the input image, x, using the relation, [3]:

$$ f_{{}_xD_m^k}\left(a\right) = \star_{q_1=1}^{M(k)} \ldots \star_{r_m=1}^{M_0} f_d(k, q_1, r_1, ..., q_m, r_m, a) \tag{7} $$

where:

$$ f_d\left(k, q_1, ..., r_m, a\right) = G\left(k, q_1, ..., r_m\right) f_x\left(G\left(k, q_1, ..., r_m\right) a\right) \tag{8} $$

and:

$$ G\left(k, q_1, r_1, ..., q_m, r_m\right) = \frac{1}{F\left(k, q_1, r_1\right) \prod_{l=2}^{m} m_0\left[q_l\right] m_0\left[r_l\right]} \tag{9} $$

where:

$$F(k,q_1,r_1)=\begin{cases} m_0[q_1]\,m_0[r_1] & for\ \ k=4\\ m_0[q_1]\,m_1[r_1] & for\ \ k=3\\ m_1[q_1]\,m_0[r_1] & for\ \ k=2\\ m_1[q_1]\,m_1[r_1] & for\ \ k=1 \end{cases} \tag{10}$$

M_0 represents the length of the impulse response $m_0, M_1$the length of m_1 and the numbers of the first two groups of convolutions in relation (7) are given by the relation:

$$M(k)=\begin{cases} M_0 & for\ \ k=4\\ M_0 & for\ \ k=3\\ M_1 & for\ \ k=2\\ M_1 & for\ \ k=1 \end{cases}\ and\ N(k)=\begin{cases} M_0 & for\ \ k=4\\ M_1 & for\ \ k=3\\ M_0 & for\ \ k=2\\ M_1 & for\ \ k=1 \end{cases} \tag{11}$$

In conformity with (7), the pdf of the wavelet coefficients is a sequence of convolutions. Hence, the random variable representing the wavelet coefficients can be written like a sum of independent random variables. So, the central limit theorem can be applied. This is the reason why the pdf of the wavelet coefficients tends asymptotically to a Gaussian, when the number of all convolutions in (7) tends to infinity. This number depends on the mother wavelets used and on the number of iterations of the DWT. For mother wavelets with a long support, this number becomes high very fast (for a small number of iterations). The mother wavelet with the shortest support is the Haar mother wavelets.

We have computed, using the relation (7) the pdf of the coefficients of the 2D DWT of an image containing a noise distributed following a log-*gamma* distribution, using the Haar mother wavelets. Because the difference between the pdfs of the wavelet coefficients obtained after the second iteration and the Gaussian is small in this case, and because the support of the mother wavelets used in practice is longer than the support of the Haar mother wavelets, used in this example, we conclude that after two iterations the pdfs of the wavelet coefficients can be considered Gaussian. For the first two iterations, heavy-tailed models must be considered. Finer analysis, measuring the distance between the real pdfs and Gaussian, are performed in [1], [2] and [3].

4 The correlation of the wavelet coefficients

The input image, x, represents, in general, the sum of the useful image, s, and of the noise image, n. Because these two random signals are not correlated, the correlation of the wavelet coefficients of the image x, is the sum of the correlations of the wavelet coefficients of the useful image and of the noise image. The correlation function of the wavelet coefficients can be computed using the following relation:

$$\Gamma_{{}_xD_m^k}[n_1,n_2,p_1,p_2]=E\left\{{}_xD_m^k[n_1,p_1]\left({}_xD_m^k[n_2,p_2]\right)^*\right\}$$

$$= \int_{R^4} E\left\{x\left(\tau_1,\tau_2\right)\right\} \cdot \psi^{k*}_{m,n_1,p_1}\left(\tau_1,\tau_2\right) \cdot \psi^{k}_{m,n_2,p_2}\left(\tau_3,\tau_4\right) d\tau_1 d\tau_2 d\tau_3 d\tau_4 \quad (12)$$

or:

$$\Gamma_{{}_xD^k_m}\left[n_1,n_2,p_1,p_2\right] = \frac{1}{4\pi^2}\int_{R^2} \gamma_x\left(2^{-m}\nu_1, 2^{-m}\nu_2\right) \cdot$$

$$\cdot \left|\alpha_2\left\{\psi^k\left(\upsilon_1,\nu_2\right)\right\}\right|^2 \cdot e^{-j\left[\upsilon_1\left(n_2-n_1\right)+\upsilon_2\left(p_2-p_1\right)\right]} d\nu_1 d\nu_2 \quad (13)$$

where the square of the absolute value representing the first factor on the last line represents the power spectral density of the one dimensional mother wavelets. If the input noise is white, with a known variance, z, it can be written:

$$\gamma_n\left(2^{-m}\nu_1, 2^{-m}\nu_2\right) = z \quad (14)$$

and the expression of the wavelet coefficients of the input noise image correlation function becomes:

$$\Gamma_{{}_nD^k_m}\left[n_1,p_1\right] = z \cdot \delta\left[n_1\right] \cdot \delta\left[p_1\right] \quad (15)$$

The same result can be obtained taking in (8) the limit for m tending to infinity. So, asymptotically the 2D DWT transforms every coloured noise into a white one. Hence this transform can be regarded as a whitening system. So, the wavelet coefficients sequences of the noise component of the input image are white noise images having the same variance. The first and second order moments of the wavelet coefficients can be computed using the following relations.

$$E\left\{{}_xD^k_m\left[n_1,p_1\right]\right\} = E\left\{\int_{R^2} x\left(\tau_1,\tau_2\right) \cdot \psi^{k*}_{m,n_1,p_1}\left(\tau_1,\tau_2\right) d\tau_1 d\tau_2\right\} = \quad (16)$$

$$= \begin{cases} 0, k = 1,2,3 \\ 2^m \cdot \mu_x, k = 4 \end{cases}$$

Only the means of the images formed with the approximation wavelet coefficients are not nulls. The mean of the DWT of the noise component of the input image is given by the relation:

$$E\left\{{}_nD^k_m\left[n_1,p_1\right]\right\} = \begin{cases} 0, k = 1,2,3 \\ -2^m \cdot \mu_n, k = 4 \end{cases} \quad (17)$$

In practice the number of iterations of the DWT is important. The dimensions of the image obtained using the approximation wavelet coefficients obtained after the last iteration are smalls. This is the reason why this image is not filtered in the denoising applications based on the use of the DWT. The variance of the wavelet coefficients of the noise component can be computed using the relation:

$$\sigma^2_{{}_x D^k_m} = E\left\{\left|{}_x D^k_m\left[n_1, p_1\right]\right|^2\right\} = \Gamma_{{}_x D^k_m}(0,0) =$$

$$= \frac{1}{4\pi^2}\int_{R^2} \gamma_x\left(2^m \nu_1, 2^m \nu_2\right) \cdot \left|\alpha_2\left\{\psi^k\left(\nu_1, \nu_2\right)\right\}\right|^2 d\nu_1 d\upsilon_2 \tag{18}$$

The DWT of the input noise component, n, has a variance given by:

$$\sigma^2_{{}_n D^k_m} = \begin{cases} z & k = 1, 2, 3 \\ z - 2^{2m}\mu_n^2 & k = 4 \end{cases} \tag{19}$$

This variance is constant for all the images formed using detail wavelet coefficients. Hence, it can be estimated using the first HH wavelet coefficients sequence. This estimation can be used for the filtering of any other detail image, formed with the detail wavelet coefficients obtained at any iteration. The correlation of the DWT of s is given by:

$$\Gamma_{{}_s D^k_m}\left[n_1, p_1\right] = 2^{2m} \cdot \Gamma_s\left[2^m n_1, 2^m p_1\right] \tag{20}$$

its mean by:

$$E\left\{{}_s D^k_m\left[n_1, p_1\right]\right\} = \begin{cases} 0, & k = 1, 2, 3 \\ 2^m \cdot \mu_s, & k = 4 \end{cases} \tag{21}$$

and its variance, by:

$$\sigma^2_{{}_s D^k_m} = 2^{2m} \cdot \sigma_s^2 \tag{22}$$

So, the variance of the detail wavelet coefficients sequences obtained starting from the useful component of the input image increases when the iteration index increases. All the relations established in this paragraph were used in [4] for the design of a denoising system for SONAR images.

5 Conclusion

A complete analysis of the 2D DWT was reported. It is proved that the 2D DWT asymptotically converges to the 2D Karhunen-Loeve transform. So, the DWT of a coloured noise with a given probability density function converges asymptotically to a white Gaussian noise. This is a generalisation of the results reported in [5]. Based on this analysis a new denoising system was built in [4]. Its performances for the treatment of the SONAR images are also reported in [4]. This statistical analysis can be used for compression purposes also. Statistical analyses of other wavelet transform will be reported soon.

REFERENCES

[1] Samuel Foucher, Gozé Bertin Bénié, Jean-Marc Boucher, "Multiscale MAP Filtering of SAR images", *IEEE Transactions on Image Processing*, vol. 10, no.1, January 2001, 49-60.

[2] Alin Achim, Panagiotis Tsakalides and Anastasios Bezerianos, "SAR Image Denoising via Bayesian Wavelet Shrinkage Based on Heavy-Tailed Modeling", *IEEE Transactions on Geoscience and Remote Sensing*, Vol. 41, No. 8, August 2003, 1773-1784.

[3] Hua Xie, Leland E. Pierce and Fawwaz T. Ulaby, "Statistical Properties of Logarithmically Transformed Speckle", *IEEE Transactions on Geoscience and Remote Sensing*, vol. 40, no. 3, March 2002, 721-727.

[4] A. Isar, S. Moga, Le débruitage des images SONAR en utilisant la transformée en ondelettes discrète à diversité enrichie, Rapport de recherche, LUSSI-TR-2004-4, Département Logiques des Usages, Sciences Sociales et Sciences de l'Information, Laboratoire Traitement Algorithmique et Matériel de la Communication, de l'Information et de la Connaissance, CNRS FRE 2658, ENST-Bretagne, 2004.

[5] A. Isar, A. Cubitchi, M. Nafornita, Algorithmes et techniques de compression, Ed.Orizonturi Universitare Timisoara,2002.