On the Discrete Wavelet Transform (DWT) Initialization Errors in Continuous-Time Applications

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Problem Statement

Designed for discrete-time signals treatment, the DWT can be used also for the processing of continuous-time signals. Such a recent application of DWT is the wavelet modulation (WM).
How to Use DWT to Process Continuous Time Signals?

Projecting the continuous time signal $x(t)$ onto the space $V_0$, defined by the scaling function $\varphi$:

$$Pn_0 \{x(t)\} = \sum_{n=-\infty}^{\infty} a_0[n] \varphi(t-n)$$

the DWT input sequence $a_0[n]$ can be obtained. This projection procedure is called DWT initialization.
The Initialization System

\[
\begin{align*}
q(t) &= q(\cdot,t) \\
\int_{-\infty}^{\infty} dt x(t) \phi^x(t) \\
\int_{-\infty}^{\infty} dt x(t) \phi^y(t - \tau) d\tau &\leq x(\tau), \phi(\tau - k) > \\
u(t) &= x(t) \phi^x(t) \\
y(t) &= \sum_{k=-\infty}^{\infty} u(k) \delta(t - k)
\end{align*}
\]

The initialization problem:
The relation between \(x[n]\) and \(a_0[n]\).

Our Goals

Establish the relation between \(x[n]\) and \(a_0[n]\).
Found the expression of the initialization error.
Establish a superior bound of the initialization error.
Application examples.
Some similar results were already obtained.
Previous Publications on the Same Topic


\[
\phi_{\alpha}^n(x_F) = n \delta_{v}^d \quad \text{Find } \alpha[n] \text{ to minimize } e[n] \text{ for } x(t) \text{ and } \phi(t) \text{ specified. The linear time-invariant solution is:}
\]

\[
\alpha[n] = F_d^{-1} \left\{ \frac{F_d \left( x(t)^* \phi^*(t) \right)_{t=n}}{F_d (x[n])} \right\}
\]
Difficulties

\[ \alpha[n] = F_d^{-1} \left\{ \frac{F_d[x(t) \ast \varphi^*(t)]_{t=n}}{F_d[x[n]]} \right\} \]

The analytical expressions of the continuous-time signal \( x(t) \) and of the scaling function \( \varphi(t) \) must be known.

The majority of scaling functions have not analytical expression.

There is a best scaling function for a specified continuous time signal \( x(t) \) and a specified application but it is not known a priori.

Two Particular Cases

- \( x(t) \in B_{\pi}^2 \)
  \[ F_d \{ \alpha[n] \} = F \{ \varphi^*(t) \} (\omega) \big|_{\omega=\Omega} \]

- \( x(t) \in B_{\pi}^2 \) and \( \varphi^*(t) \in B_{\pi}^2 \)
  \[ \alpha[n] = \varphi^*[n] \]

The systems with impulse responses \( \alpha[n] \) and \( \varphi(t) \) are equivalent on the base of the impulse invariance method.

This relation was already proposed in [6] and [7] (without derivation).
The Initialization Error

\[ \alpha[n] = F^{-1} \left\{ \frac{F_d(x(t)^* \varphi^{*\tau}(t)_{t=n})}{F_d(x[n])} \right\} \]

\[ \epsilon[n] = \langle x(\tau), \varphi^{*\tau}(n-\tau) \rangle_{L^2} - \langle x[k], \alpha^*[n-k] \rangle_{L^2} \]

In the second particular case the initialization error vanish.

A Superior Bound of the Initialization Error

\[ |e[n]| \leq \langle x(\tau), \varphi'(n-\tau) \rangle_{L^2} + |\langle x[k], \alpha^*[n-k] \rangle_{L^2}| = \text{supb}[n] \]
Two Extreme Examples

1. \[ x(t) = \text{sinc}(\pi t) \]
   \[ x(t) \in B_\pi^2 \]

If: \[ \varphi(t) = \text{sinc}(\pi t) \] then the hypotheses of the second particular case are obtained and:

\[ a[n] = \delta[n] \]
\[ a_0[n] = \text{sinc}(\pi n) \]
\[ e[n] = 0 \]

There is no need of initialization in this case.

Second Example

2. \[ x(t) = \sigma(t - \frac{9}{10}) - \sigma(t - \frac{11}{10}) \]

If the Haar scaling function is selected: \[ \varphi(t) = \sigma(t) - \sigma(t - 1) \], then:

\[ a[n] = \frac{1}{10} \delta[n-1] + \frac{1}{10} \delta[n-2]; \quad a_0[n] = \frac{1}{10} (\delta[n-2] + \delta[n-3]) \]
\[ e[n] = \frac{1}{10} (\delta[n] + \delta[n-1] - \delta[n-2] - \delta[n-3]) \]
\[ sup[n] = \|e[n]\| = \frac{1}{10} (\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]) \]

The initialization error is higher than the response \[ a_0[n] \]. The energy of the sequence \[ sup[n] \] is the double of the energy of the sequence \[ a_0[n] \].

The initialization is very difficult in this case.
Other Solutions


Wavelet Modulation

The initialization problem appears at the output of the communication channel. It cannot be solved completely because the analytical expression of the signal $x(t)$ is not known. More, this signal represents a mixture of a deterministic and a random component. The maximum value of $x(t)$ can be estimated by simulations. Using it in conjunction with the proposed superior bound formula, we intent to appreciate the initialization error for the wavelet modulation in the future.