On the Discrete Wavelet Transform (DWT) Initialization Errors in Continuous-Time Applications

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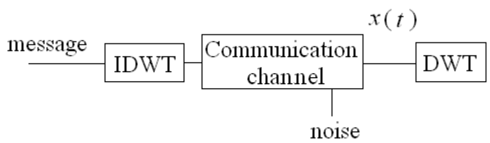
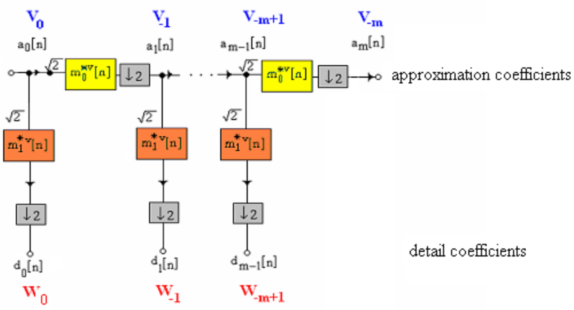
Problem Statement

Designed for discrete-time signals treatment, the DWT can be used also for the processing of continuous-time signals.

Such a recent application of DWT is the wavelet modulation (WM).

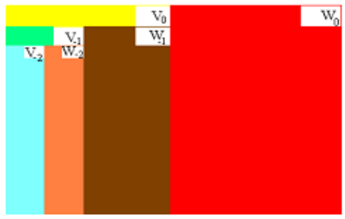


DWT



$$DWT\{a_0[n]\} = [a_m[n], d_{m-1}[n], \ldots, d_0[n]]$$

Multiresolution Analysis



$$V_0 = V_{-m} \overset{\perp}{\oplus} \left(\bigcup_{1 \le p \le m-1} W_{-p} \right)$$

$$Pr_{V_0}\{x(t)\} = \sum_{n=-\infty} a_0[n]\varphi(t-n);$$

$$Pr_{V_{-1}}\{x(t)\} = \sum_{n=-\infty}^{\infty} a_1[n]2^{-\frac{1}{2}}\varphi(2^{-1}t-n);$$

$$Pr_{W_{-1}}\{x(t)\} = \sum_{n=-\infty}^{\infty} d_1[n]2^{-\frac{1}{2}}\psi(2^{-1}t-n);$$

...

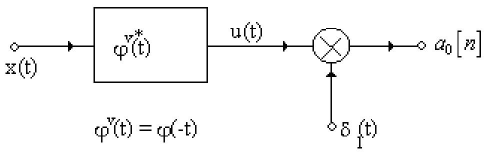
How to Use DWT to Process Continuous Time Signals ?

Projecting the continuous time signal $x(t)$ onto the space V_0, defined by the scaling function φ:

$$Pr_{V_0}\{x(t)\} = \sum_{n=-\infty}^{\infty} a_0[n]\varphi(t-n)$$

the DWT input sequence $a_0[n]$ can be obtained. This projection procedure is called DWT initialization.

The Initialization System



$$u(t) = x(t) * \varphi^{v*}(t) = \int_{-\infty}^{\infty} x(\tau)\varphi^{v*}(t-\tau)d\tau$$

$$y(t) = \sum_{k=-\infty}^{\infty} u(k)\delta(t-k)$$

$$u(k) = \int_{-\infty}^{\infty} x(\tau)\varphi^{*}(\tau-k)d\tau = < x(\tau), \varphi(\tau-k) >$$

$$u(k) = a_0[k]$$

The initialization problem:

The relation between $x[n]$ and $a_0[n]$.

Our Goals

Establish the relation between $x[n]$ and $a_0[n]$.

Found the expression of the initialization error.

Establish a superior bound of the initialization error.

Application examples.

Some similar results were already obtained.

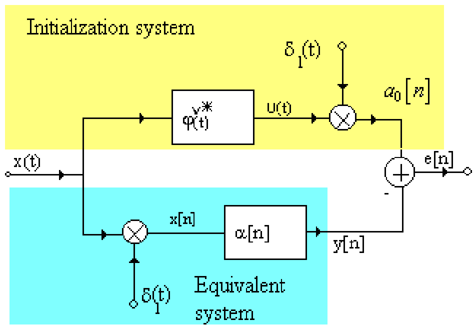
Previous Publications on the Same Topic

[3] M. J. Shensa, "The Discrete Wavelet Transform: Wedding the A Trous and Mallat Algorithms", *IEEE Transactions on Signal Processing*, vol. 40, no.10, October 1992, pp.2464-2482,

[6] O. Rioul, P. Duhamel, "Fast Algorithms for Discrete and Continuous Wavelet Transforms", *IEEE Trans. Info. Theory*, vol. 38, no.2, March 1992, pp.569-586,

[7] P. Abry, P. Flandrin, "On the Initialization of the Discrete Wavelet Transform Algorithm", *IEEE Signal Processing Letters,* vol.1, no.2, February 1994, pp.32-34.

The Initialization Problem



Find $\alpha[n]$ to minimize $e[n]$ for $x(t)$ and $\varphi(t)$ specified. The linear time-invariant solution is:

$$\alpha[n] = F_d^{-1}\left\{ \frac{F_d\left(x(t) * \varphi^{\nu *}(t)|_{t=n}\right)}{F_d\left(x[n]\right)} \right\}$$

Difficulties

$$\alpha[n] = F_d^{-1}\left\{\frac{F_d\left(x(t) * \varphi^{v*}(t)|_{t=n}\right)}{F_d\left(x[n]\right)}\right\}$$

The analytical expressions of the continuous-time signal $x(t)$ and of the scaling function $\varphi(t)$ must be known.

The majority of scaling functions have not analytical expression.

There is a best scaling function for a specified continuous time signal $x(t)$ and a specified application but it is not known a priori.

Two Particular Cases

- $x(t) \in B_\pi^2$

$$F_d\{\alpha[n]\} = F\{\varphi^{v*}(t)\}(\omega) 1_{[-\pi,\pi]}(\omega)|_{\omega=\Omega}$$

- $x(t) \in B_\pi^2$ and $\varphi^{v*}(t) \in B_\pi^2$

$$\alpha[n] = \varphi^{v*}[n]$$

The systems with impulse responses $\alpha[n]$ and $\varphi(t)$ are equivalent on the base of the impulse invariance method.

This relation was already proposed in [6] and [7] (without derivation).

The Initialization Error

$$\alpha[n] = F_d^{-1}\left\{ \frac{F_d\left(x(t) * \varphi^{v*}(t)|_{t=n}\right)}{F_d\left(x[n]\right)} \right\}$$

$$e[n] = \left\langle x(\tau), \varphi^{v*}(n-\tau) \right\rangle_{L^2} - < x[k], \alpha^*[n - k] >_{l^2}$$

In the second particular case the initialization error vanish.

A Superior Bound of the Initialization Error

$$|e[n]| \leq |< x(\tau), \varphi^v(n-\tau) >_{L^2}| + |< x[k],\ \alpha^*[n-k] >_{l^2}| = sup6[n]$$

Two Extreme Examples

1. $_1x(t) = sinc(\pi t)$

$_1x(t) \in B_\pi^2$

If: $_1\varphi(t) = sinc(\pi t)$ then the hypotheses of the second particular case are obtained and:

$$_1\alpha[n] = \delta[n]$$

$$_1a_0[n] = sinc(\pi n)$$

$$_1e[n] = 0$$

There is no need of initialization in this case.

Second Example

2. $_2x(t) = \sigma(t - \frac{9}{10}) - \sigma(t - \frac{11}{10})$

If the Haar scaling function is selected: $_2\varphi(t) = \sigma(t) - \sigma(t-1)$, then:

$$_2\alpha[n] = \frac{1}{10}\delta[n-1] + \frac{1}{10}\delta[n-2]\,; \quad _2a_0[n] = \frac{1}{10}(\delta[n-2] + \delta[n-3])$$

$$e[n] = \frac{1}{10}(\delta[n] + \delta[n-1] - \delta[n-2] - \delta[n-3])$$

$$_2sup\mathit{б}[n] = |e[n]| = \frac{1}{10}(\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3])$$

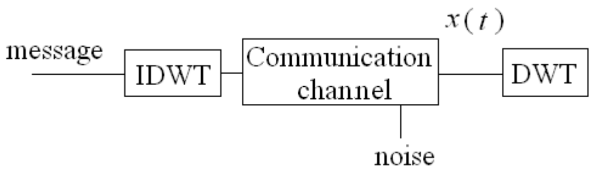
The initialization error is higher than the response $_2a_0[n]$. The energy of the sequence $_2sup\mathit{б}[n]$ is the double of the energy of the sequence $_2a_0[n]$.

The initialization is very difficult in this case.

Other Solutions

Adaptive Filtering: X. G. Xia, C. C. J. Kuo, Z. Zhang, "Wavelet coefficient computation with optimal prefiltering", *IEEE Transactions on Signal Processing*, vol.42, no. 8, Aug. 1994, pp.2191-2197.

Wavelet Modulation



The initialization problem appears at the output of the communication channel. It can not be solved completely because the analytical expression of the signal $x(t)$ is not known. More, this signal represents a mixture of a deterministic and a random components. The maximum value of $x(t)$ can be estimated by simulations. Using it in conjunction with the proposed superior bound formula, we intent to appreciate the initialization error for the wavelet modulation in the future.