Buletinul Ştiinţific al Universităţii "Politehnica" din Timişoara

Seria ELECTRONICĂ şi TELECOMUNICAŢII
TRANSACTIONS on ELECTRONICS and COMMUNICATIONS

Tom 53(67), Fascicola 1-2, 2008

On the Discrete Wavelet Transform Initialization Errors in Continuous-Time Applications

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Abstract – The discrete wavelet transform can be used to process continuous-time signals. To use it, the initialization errors must be minimized. This is the aim of this paper. The results are justified and are presented in a unitary manner. We give a strategy to accomplish this minimization. Some examples are presented. A superior bound of these errors is also presented.

Keywords: discrete wavelet transform, initialization, continuous-time applications

I. INTRODUCTION

A modern problem in signal processing theory is the analysis of non-stationary signals. The tools for this analysis are the time-frequency representations. One of the most important time-frequency representations is the Continuous Wavelet Transform (C.W.T), [1].
Given a non-stationary signal $x(t)$, wavelet transform consists of computing coefficients that are inner products of the signal and a family of "wavelets". The wavelet corresponding to scale a and time-location b is:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right)$$

where $\psi(t)$ is the mother wavelets. The CWT of $x(t)$ is:

$$CWT_x(a,b) = \int_{-\infty}^{\infty} x(t)\psi^*_{a,b}(t)dt$$

Time t and time-scale parameters vary continuously. This transform can be discretized. If the time remains continuous but time-scale parameters (b,a) are sampled on a "dyadic" grid, then the wavelet series coefficients are obtained:

$$C_{j,k} = CWT_x\left(2^j, k\,2^j\right) \qquad (1)$$

Let Vj, $j\in Z$, be a multiresolution analysis of $L^2(R)$, [2], generated by the scaling function $\varphi(t)$. Let $x_j(t)$, $j\geq 0$, be the projection of the finite energy signal $x(t)$ on the space V_j. The signal $x_j(t)$, $j>0$, represents the approximation of resolution j of the signal $x_0(t)$. The coefficients of the development of $x_j(t)$ in the basis of V_j are: $<x_0(t), \varphi_{j,n}(t)> = b_{j,n}$. Computation of these coefficients are simplified due to the following recurrence relation:

$$b_{j,n} = \sum_p b_{j-1,p}\, h^*_{2n-p}$$

where:

$$h_{2n-l} = <\varphi_{1,n}(t), \varphi(t-l)> \qquad (2)$$

[2]. The approximation of $x_0(t)$ by $x_j(t)$ is realized with the error $e_j(t)=x_0(t)-x_j(t)$. For every multiresolution analysis of $L^2(R)$, V_j, $j\in Z$, a correspondent orthogonal decomposition of $L^2(R)$, W_j $j\in Z$ can be built. The error $e_j(t)$ is the projection of the signal $x_0(t)$ on the space W_j, $j>0$. It can be written:

$$x_0(t) = x_J(t) + \sum_{j=1}^{J} e_j(t)$$

The coefficients $c_{j,k} = <x_0(t), \psi_{j,k}(t)>$ can be computed with the following recurrence relation:

$$c_{j,k} = \sum_l b_{j-1,l}\, g^*_{2n-l}$$

where:

$$g_{2n-l} = <\psi_{1,n}(t), \varphi(t-l)> \qquad (3)$$

[2]. So, using the coefficients $b_{0,n}$ of the signal $x_0(t)$, the $x_J(t)$ and $e_j(t)$, $j=1,\ldots,J$ can be obtained. The transformation of the discrete signal $b_{0,k}$ in the sequence of signals: $c_{1,k}$, $c_{2,k}$, ..., $c_{J,k}$, $b_{J,k}$ is named Discrete Wavelet Transform (DWT). The main problem of this paper is to obtain the signal $x_0(t)$ (its coefficients $b_{0,k}$) the projection of $x(t)$ on the space V_0, starting from a known signal $x(t)$. This operation represents the initialization of the DWT. The signal

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$x_0(t)$ can be exactly reconstructed after the application of the DWT and the inverse DWT to the sequence $b_{0,k}$. Because this signal represents the projection of the signal $x(t)$ on the space V_0 it can be computed using a projection filter. This is the reason why, in the reconstruction phase, the signal $x(t)$ can be generated using the signal $x_0(t)$ (that can be obtained after the application of the inverse DWT) and the inverse system corresponding to the projection filter.

II. THE INITIALIZATION PROBLEM

For the use of an algorithm for the processing of a continuous-time signal, $x(t)$, this signal must be sampled, obtaining the sequence $x[n]$. The initialization of the DWT consists in the computation of the sequence $b_{0,k}$ starting from the sequence $x[n]$. The DWT is a very fast transformation. It is faster then the FFT. The problem is to find the impulse response of a discrete-time system, $\alpha[n]$, such that its response $y[n]$, to the input signal $x[n]$, to be a good approximation of the signal $b_{0,n}$. This is a very interesting problem, pointed out in [3], [4], [6] and [7].

II. 1 How to obtain the sequence $b_{0,n}$ starting from the signal x(t) ?

The system in figure 1 transforms the signal $x(t)$ into the sequence $b_{0,n}$.

The expression of the signal $u(t)$, in figure 1, is:

$$u(t) = x(t) * \varphi^{v*}(t) = \int_{-\infty}^{\infty} x(\tau)\varphi^{v*}(t-\tau)d\tau$$

Sampling this signal we obtain:

$$y(t) = \sum_{k=-\infty}^{\infty} u(k)\delta(t-k)$$

The samples of $u(t)$ have the values:

$$u(k) = \int_{-\infty}^{\infty} x(\tau)\varphi^{*}(\tau-k)d\tau = \langle x(\tau), \varphi(\tau-k) \rangle$$

So:

$$u(k) = b_{0,k}$$

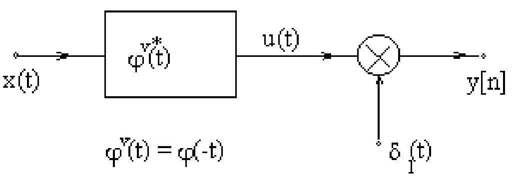

Figure 1. The transformation of the signal $x(t)$ into the sequence $b_{0,n}$.

Hence the system in figure 1 transforms the signal $x(t)$ into the sequence $b_{0,k}$.

II.2 The Computation of the Impulse Response α[n]

The initialization problem is presented in figure 2. The condition to cancel the error $e[n]$ is:

$$F_d(x[n])F_d(\alpha[n]) = F_d\left(\left(x(t) * \varphi^{v^*}(t)\right)\Big|_{t=n}\right)$$

where F_d represents the Fourier transform in discrete-time. So the expression of the impulse response $\alpha[n]$ is:

$$\alpha[n] = F_d^{-1}\left\{\frac{F_d\left(x(t) * \varphi^{v*}(t)|_{t=n}\right)}{F_d(x[n])}\right\} \quad (4)$$

Shensa proposed this exact solution of the DWT initialization problem, too, without derivation, in [3], (relation (4.12.c)). Unfortunately this relation can not be applied when the expression of the signal $x(t)$ is unknown.

The continuous in time convolution requested in (4) can not be computed with a computer. In the following, two particularizations of (4), more useful in practice, are presented.

- *Case I*

When the signal $x(t)$ is band-limited, $x(t) \in B_\pi^2$, the relation (4) becomes:

$$\alpha_a[n] = z[n] \quad (5)$$

The Fourier transform of the signal $z[n]$ is:

$$F_d\{z[n]\} = F\{\varphi^{v*}(t)\}(\omega)\, I_{[-\pi,\pi]}(\omega)|_{\omega=\Omega}$$

where F represents the Fourier transform for continuous-time signals. This is so because the continuous-time signal with the Fourier transform $F\{\varphi^{v*}(t)\}(\omega)\ I_{[-\pi,\pi]}(\omega)$ is a band-limited one, like $x(t)$. This is the reason why their continuous convolution is also a member of the B_π^2 space. But, for such signals, the Fourier transform in discrete-time is identical in the interval [-π,π] with their Fourier transform in continuous time.

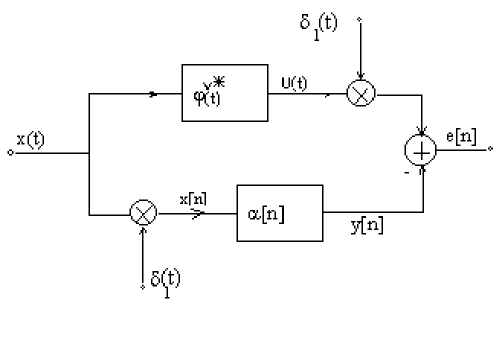

Figure 2. The initialization problem.

Case II. To the hypothesis of case I is added the supplementary hypothesis that $\varphi^{v*}(t)\in B_{\pi}^{2}$. In this case the relation (4) becomes (see the relation (5) and the case I):

$$\alpha_a[n]=\varphi^{v*}[n] \qquad (6)$$

So, the system whit the impulse response $\alpha[n]$ is equivalent with the system with impulse response $\varphi^{v*}(t)$, on the base of the impulse invariance method, [5]. This approximation of the solution of the initialization problem, without derivation, is proposed in [6] (relation (15) for $\chi(t)=\delta(t)$) and in [7] (relation (14)).

III. THE ESTIMATION OF THE APPROXIMATION ERROR

At the beginning are presented some preliminary results.

P1. The discrete-time signal obtained by uniform sampling with a unitary step of a finite energy signal has finite energy.

P2 The convolution of two finite energy continuous-time signals is a new finite energy signal.

P3 The convolution of two finite energy discrete-time signals is a new finite energy discrete-time signal.

Now, the approximation errors for the initialization methods in relation (5) and (6) can be estimated. The initialization error is:

$$\begin{aligned} &e[n]=u[n]-v[n] \text{ where } u[n]= \\ &(x(t)*\varphi^{v*}(t))|_{t=n} \overset{P1,P2}{=} \\ &=<x(\tau),\varphi^{v}(n-\tau)>_{L^2} \qquad (7) \\ &v[n]=x[n]*\alpha_a[n]= \\ &\overset{P3}{=}<x[k],\alpha_a^{*}[n-k]>_{l^2} \end{aligned}$$

A superior bound of this error can be also estimated. Indeed:

$$|e[n]|\leq|<x(\tau),\varphi^{v}(n-\tau)>_{L^2}|+|<x[k],\alpha_a^{*}[n-k]>_{l^2}| \quad (8)$$

Because there is an isomorphism between the spaces B_{π}^{2} and l^2 introduced by sampling, we can observe that the error $e[n]$ is zero when the functions $x(t)$ and $\varphi(t)$ are elements of B_{π}^{2}. A different strategy for the decreasing of the initialization error was considered in [4]. This strategy is based on adaptive filtering. Its advantage is the fact that the analytical expression of the corresponding scaling function is not requested. The disadvantage is the amount of computation requested by the adaptive filtering procedure. Unfortunately the authors of this article have not derived a superior bound for the initialization error obtained using their strategy. In the following we present some examples to compare the precision of different initialization methods.

IV. SOME EXAMPLES

A. *Example 1*

Let:

$$_{1}x(t)=sinc(\pi t)$$

Because this is a band-limited signal, the appropriate scaling function for the computation of its discrete wavelet transform is:

$$_{1}\varphi(t)=sinc(\pi t)$$

because this is also a band-limited signal. Using the relation (4) it can be written:

$$_{1}\alpha[n]=\delta[n]$$

Hence:

$$_{1}b_{0,n}=sinc(\pi n)$$

Using the relation (7), the initialization error can be computed. It is equal with zero at any moment. So the initialization procedure is exact in this case.

For this example the other initialization methods (described in the relations (5) and (6)) are also exact. Unfortunately the scaling function chosen in this example has not a compact support.

B. *Example 2*

Let:

$$_{2}x(t)=\sigma(t-\frac{9}{10})-\sigma(t-\frac{11}{10})$$

Tacking into account the waveform of this signal, the appropriate wavelet for its analysis, is the Haar wavelet.

So:

$$_{2}\varphi(t)=\sigma(t)-\sigma(t-1)$$

Computing the convolution in (4) we obtain:

$$_{2}x(t)*_{2}\varphi^{v*}(t)\Big|_{t=n}=\frac{1}{10}\delta[n]+\frac{1}{10}\delta[n-1]$$

But:

$$_{2}x[n]=\delta[n-1]$$

So, using (4) we obtain:

$$_{2}\alpha[n]=\frac{1}{10}\delta[n-1]+\frac{1}{10}\delta[n-2]$$

and:

$$_{2}b_{0,n}={}_{2}x[n]*_{2}\alpha[n]=\frac{1}{10}(\delta[n-2]+\delta[n-3])$$

The initialization proposed in (4) is not exact because in this example the function $\varphi(t)$ is not band-limited (the initialization error can be computed using the relation (7)) and the initialization proposed in (6) can not be applied because the function $\varphi(t)$ can not be sampled (this is not a continuous function).

V. CONCLUSION

The use of DWT is a very elegant solution for the digital processing of a continuous-time signal. In this purpose it must be initialized. The initialization problem was formulated 14 years ago, but its importance is highlighted by some new applications of the DWT, like for example the wavelet modulation, [8]. In this case the continuous-time signal obtained at the output of the communication channel must be demodulated using the DWT. So, the wavelet demodulation is an example of application of the DWT for the processing of continuous-time signals.

The initial sequence $b_{0,n}$ represents the result of the convolution between the sampled version of the analyzed signal $x[n]$ and an impulse response $\alpha[n]$. The exact expression of this impulse response is presented in relation (4).

Unfortunately is difficult to apply this solution in practice, because the convolution of two continuous-time signals can not be exactly computed with numerical algorithms and because there are occasions when one or both expressions of the signal $x(t)$ and of the scaling function are not known. This is the reason why approximations for the expression of the impulse response are presented too, in relation (5) and (6). The expression in relation (6) supposes the construction of the initialization filter with the aid of a method for the equivalence of a continuous- time with a discrete-time system. In fact the equivalence based on the invariance of the impulse response is used. Of course there are also other equivalence methods. So other initialization filters, similar to the one presented in relation (6) can be built using the equivalence method based on the approximation of a differential equation with an equation with finite differences or using the equivalence method based on the bilinear transform. A priori, the better one is that based on the bilinear transform.

The expression of the impulse response $\alpha[n]$ depends on the scaling function $\varphi(t)$ that generates the specified space V_0. This is the reason why we have chosen in the examples presented the most appropriate scaling functions. Unfortunately there are some wavelet mothers with corresponding scaling functions without analytical expression (this is the case for the compact support wavelets introduced by Ingrid Daubechies in [2] for example). In this paper are estimated for the first time the errors occurring in the initialization process when the relation (5) and (6) are used in the two examples presented. A superior bound of these errors is also presented in relation (8). Unfortunatelly its practical use is limited due to the fact that in practice the expression of the continuous-time signal and of the scaling function are not known. Any-way it can be used for calibration purposes.

ACKNOWLEDGEMENT

This research was realized in the framework of the CNCSIS Grant number 351.

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