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On the Encoding of the Multi-Non-Binary Convolutional Codes

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Abstract - Recently, Douillard et al. proposed a new family of multi-binary turbo-codes based on the parallel concatenation of two constituent convolutional codes with multiple inputs that has better global performance than classical turbo-codes. The encoder is based on an *r*-inputs linear feedback shift register (LFSR). In this paper, we show that the encoder can also be represented by the observer canonical configuration. This configuration is essential to reduce the computational complexity of the code design procedure especially for moderate codeword sizes. Indeed, in this context, an exhaustive search usually provides better results than the EXIT chart or similar optimization tools. We show that the second configuration reduces the computational complexity of the search up to 300%. Based on this strategy, we were able to design a rate-1/2 turbo-code with two inputs and memory m=3 which outperforms the turbo-code with same characteristics proposed by Douillard et al. by 0.25 decibels for a frame error rate of 10^{-4}.

Keywords: recursive and systematic multi-binary convolutional code, generator matrix, turbo-code.

I. INTRODUCTION

The multi-binary turbo-codes (MBTC) also referred as non-binary turbo-codes ([1]-[2]) have several advantages compare to the initial binary turbo-codes in [3] like faster convergence and lower error floor. We refer to [4] for a more detailed analysis. The design of the MBTC generally requires an exhaustive search through large sets of feasible codes [5]-[6] and interleavers [7]. This is particularly true for moderate codeword lengths. In that case, the asymptotic threshold determined by EXIT chart or by similar optimization tools may not be accurate due to the lack of randomness of the structure of the interleaver and an exhaustive search provides often better results. In the case of turbo-codes with multiple binary and non-binary inputs (MBTC [8] and MNBTC [9], respectively), this search is a major burden for their optimization since the computational complexity of the search is exponential with the number of inputs *r*. Therefore, restricting the search to a limited set of `good' M(N)BTC is essential.

The MBTC proposed in [8] consist of the parallel concatenation of two rate-$r/(r+1)$ recursive systematic convolutional codes (RSC) where *r* represents the number of inputs of the code. The encoders are based on multiple-input linear feedback shift registers (LFSR). An alternative encoding method for M(N)BTC consists in a generalization of the Fibonacci encoder ([10]-[14]). Whereas both encoding methods are equivalent for $r=1$, i.e., for the binary single input turbo-codes with and without puncturing, they give two different sets of encoder in the case of multiple inputs ($r>1$). In this paper, we compare both sets according several criteria commonly used in the system theory: cardinal number of both sets, element-to-element equivalence between both sets. This comparison is important in order to select the best encoding method for the design of the MBTC and MNBTC based on an exhaustive search.

The rest of the paper is organized as follows. Section II describes the encoding method proposed in [8]. In the third section, an alternative encoding method based on the generalization of a Fibonacci encoder is presented. In Section IV, the conditions of equivalence between both methods are determined. Finally, in Section 5, we present our optimization results and compare the packet error rate performance with the MBTC from [1].

II. CANONICAL FORM OF CONVOLUTIONAL ENCODERS BASED ON LINEAR FEEDBACK SHIFT REGISTER

Fig.1 shows the general structure of a multiple-input recursive systematic convolutional encoder that is used in [8] for each of both constituent codes. The encoder is based on an *r*-input linear feedback shift register (LFSR). This encoder is generally not decomposable into *r* single-input encoders, i.e., it cannot be represented by the controller canonical

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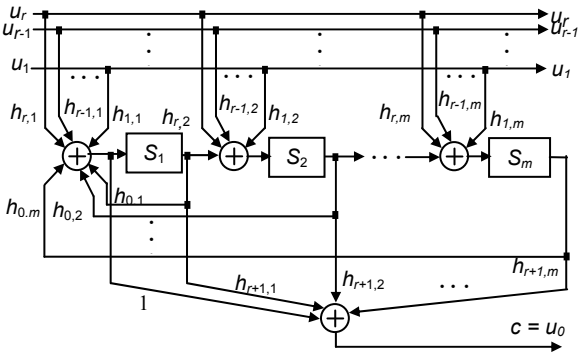


Fig.1 Canonical configuration of a multiple-input rate-$r/(r+1)$ RSC convolutional encoder based on linear feedback shift register.

configuration [1]. Throughout the paper, in order to simplify the notations, this configuration is simply referred as the canonical form of type 'H'.

The encoder has r inputs $u_1, u_2, \ldots u_r$ and $r+1$ outputs corresponding to the r inputs and one redundant bit u_0 also referred as c. The current encoder state is given by the outputs of the m shift registers $s_1, s_2, \ldots s_m$. The inputs u_i, $i=1 \ldots r$ are physically connected to the j-th adder if $h_{ij}=1$; $S_t = [\, s_m^t \ldots s_2^t \; s_1^t \,]^T$ and $U_t = [\, u_r^t \; u_{r-1}^t \ldots \; u_1^t \,]^T$ describe the encoder state at the time t and the input vector of size $r \times 1$, respectively where x^T denotes the transpose of the vector x. The input/current state and output/current state relations of the encoder at the time t can be expressed in the compact form:

$$S_{t+1} = H_0 \cdot U_t + T \cdot S_t \tag{1}$$

$$c^t = H_E \cdot S_t + W \cdot S_{t+} \tag{2}$$

where the generator matrix H_0 and the matrix T are defined as follows:

$$H_0 = \begin{bmatrix} h_{r,m} & \ldots & h_{1,m} \\ \ldots & \ldots & \ldots \\ h_{r,2} & \ldots & h_{1,2} \\ h_{r,1} & \ldots & h_{1,1} \end{bmatrix} \tag{3}$$

and

$$T = \begin{bmatrix} 0_{(m-1)\times 1} & I_{m-1} \\ & H_R \end{bmatrix} \tag{4}$$

with

$$H_R = [h_{0,m} \; \ldots h_{0,2} \; h_{0,1} \tag{5}$$

In order to have a decodable code [10], we assume that H_0 is full rank. In (2), the vector W is equal to $[0\ 0 \ldots 0\ 1]_{1\times m}$, and:

$$H_E = [h_{r+1,m} \; \ldots h_{r+1,2} \; h_{r+1,1}]. \tag{6}$$

In order to compare this canonical form based on LSFR with the observer canonical form presented in next section, we determine the transfer function matrix of the encoder. Let define the Laurent series of the sequence x_t as $X(D) = \sum_{t=-\infty}^{\infty} x_t \cdot D^t$. Assuming that the matrix $I_m + D \cdot T$ is invertible, i.e. $det(I_m + D \cdot T) = h_0(D) \neq 0$ with $h_i(D) = \sum_{j=1}^{m} h_{ij} \cdot D^j + 1$, the transfer function which corresponds to the redundant sequence $C(D)$ of the encoder and the global transfer function matrix of the code $M_h(D)$ are respectively equal to:

$$C(D) = M_h(D) \cdot U(D) \tag{7}$$

$$M_h(D) = D \cdot H_E \cdot (I_m + D \cdot T)^{-1} \cdot H_0 + W \cdot (I_m + D \cdot T)^{-1} \cdot H_0 \tag{8}$$

After some algebraic manipulations, it can be shown that:

$$(I_m + D \cdot T)^{-1} = \frac{1}{b(D)} \cdot P_m^T(D) \cdot H_R \cdot \Delta_m(D) + D^{-1} \cdot \Delta_m(D)$$

where $P_m(D) = [D^{m-1} \; \ldots D \; 1]$ and $\Delta_m(D)$ is a Toeplitz matrix with first row $[D \; D^2 \; \ldots \; D^{m-1} \; D^m]$ and first column $[D \; 0 \ldots 0 \; 0]^T$. Therefore, the transfer function matrix $M_h(D)$ can be simplified as:

$$M_h(D) = \left(\frac{h_{r+1}(D)}{h_0(D)} \cdot H_R + H_E \right) \cdot \Delta_m(D) \cdot H_0 + [h_{r,1} \ldots h_{2,1} \; h_{1,1}] \tag{9}$$

In the next section, we introduce a second canonical form for multiple-input encoders based on Fibonacci representation.

III. OBSERVER CANONICAL FORM OF CONVOLUTIONAL ENCODERS WITH MULTIPLE INPUTS

A recursive and systematic convolutional encoder with multiple inputs (MIRSC) is generally not decomposable into r single-input encoders, i.e., this encoder can generally not be represented by an equivalent structure with one shift register for each input. However, we show in this section that a realizable structure consists to have one shift register for the single output c as shown in Fig. 2. In system theory, this canonical form is referred as the *observer canonical form*, [10]. Throughout the paper, for sake of simplicity, we refer this scheme as the canonical form of type 'G'.

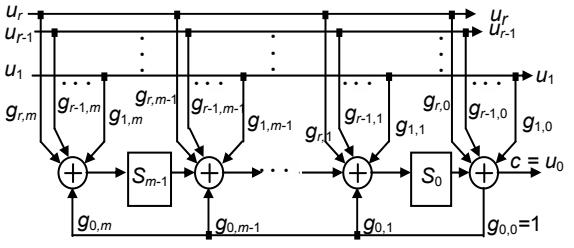


Fig.2 Observer canonical configuration of a multiple-input rate-$r/(r+1)$ RSC convolutional encoder.

Let $S_t = [\, s^t_{m-1}\ s^t_{m-2}\ \dots\ s^t_0\,]^T$ and $U_t = [\, u^t_r\ u^t_{r-1}\ \dots\ u^t_1\,]^T$ denote the m-component column vector describing the encoder state at the time t and the input vector of size $r\times 1$, respectively. The input/current state and output/current state relations of the encoder at the time t can be expressed in the compact form:

$$S_{t+1} = G_T \cdot U_t + T^T \cdot S_t \qquad (10)$$

$$c^t = G_L \cdot U_t + W \cdot S_t \qquad (11)$$

where G_T and T^T are defined as:

$$G_T = G_0 + G_R \cdot G_L \qquad (12)$$

$$T^T = \left[\begin{array}{c|c} 0_{1\times m-1} & G_R \\ \hline I_{m-1} & \end{array}\right], \qquad (13)$$

G_L and W are equal to $[g_{r,0}\ g_{r-1,0}\ \dots\ g_{1,0}]$ and $[0\ 0\ \dots\ 0\ 1]_{1\times m}$, respectively. In order to have a decodable code, we assume that G_T is full rank [10]. Moreover, G_0 is defined as:

$$G_0 = \begin{bmatrix} g_{r,m} & g_{r-1,m} & \cdots & g_{1,m} \\ g_{r,m-1} & g_{r-1,m-1} & \cdots & g_{1,m-1} \\ \cdots & \cdots & \cdots & \cdots \\ g_{r,1} & g_{r-1,1} & \cdots & g_{1,1} \end{bmatrix} \qquad (14)$$

and $G_R = [g_{0,m}\ g_{0,m-1}\ \dots\ g_{0,1}]^T$. After some algebraic manipulations and by defining $g_0(D)$ as $g_0(D) = \sum_{j=0}^{m} g_{0j} \cdot D^j$, it can be shown that the transfer function $C(D)$ of redundant sequence and the transfer function matrix $M_g(D)$ of the encoder are respectively equal to:

$$C(D) = G_L \cdot U(D) + W \cdot S(D) \qquad (15)$$

$$M_g(D) = \frac{1}{g_0(D)} \cdot \left(D \cdot P_m(D) \cdot G_0 + G_L\right) \qquad (16)$$

After deriving the transfer function matrices $M_h(D)$ and $M_g(D)$ for both canonical forms H and G in (9) and (16) respectively, we investigate in the next section the relations of equivalence between the canonical forms H and G.

IV. EQUIVALENCE BETWEEN ENCODERS DEFINED WITH THE CANONICAL FORMS H AND G

First, we define the relation of equivalence between the canonical forms H and G:

Definition 1: The canonical forms H and G are equivalent if for any data input $U(D)$, both canonical forms give the same output $C(D)$, i.e.:

$$M_g(D) = M_h(D) \qquad (17)$$

where $M_h(D)$ and $M_g(D)$ are defined in (9) and (16), respectively.

A. *Single input classical binary case (r=1 and g_{10}=1):*

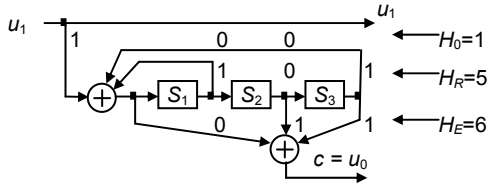
We first investigate the equivalence between both methods in the single input case and when $g_{10}=1$ as in the classical case [3].

Theorem 1 ([12] p.1220): In the case of single input (r=1), both configurations are equivalent.

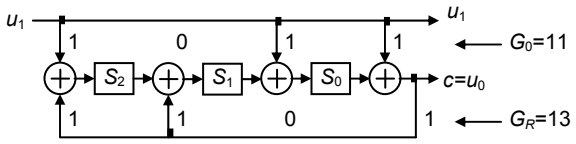
Proof: For r=1, $H_0=W^T$. By multiplying both sides of (10) by $h_0(D)$, we have:

$$h_0(D)\cdot M_h(D) = (h_2(D)\cdot H_R + h_0(D)\cdot H_E)\cdot D\cdot P_m^T(D) + h_0(D)$$
$$= h_2(D)\cdot(h_0(D)+1) + h_0(D)\cdot(h_2(D)+1) + h_0(D) = h_2(D).$$

Since G_0 is a column vector and G_L is equal to 1 in the single input case, (16) can be written as: $g_0(D)\cdot M_g(D) = g_{1,m}\cdot D^m + \dots + g_{1,2}\cdot D^2 + g_{1,1}\cdot D + 1 = g_1(D)$. So, if $h_2(D) = g_1(D)$ and $h_0(D) = g_0(D)$ both configurations H and G for the classical binary case are one-to-one equivalent.



a) Canonical configuration for the encoder defined by H=[6 1 5].



b) Observer canonical form for the encoder defined by G=[11 13].

Fig.3 The two canonical forms for the $13/15_8$ convolutional RSC

We illustrate this result with the following example. Suppose the RSC with the output and feedback polynomials equal to $15_8=1+D^2+D^3$ and $13_8=1+D+D^3$, respectively. By defining the full generator matrices H and G as follows:

$$H = [\, H_E^T\ H_0\ H_R^T\,] = [h_{r+1}\ h_r \dots h_1\ h_0]_{10} \qquad (18)$$

and

$$G = \begin{bmatrix} G_0 & G_R \\ G_L & 1 \end{bmatrix} = [g_r \dots g_1\ g_0]_{10}, \qquad (19)$$

H and G are equal to [6 1 5] and [13 11] in this example. The two canonical forms H and G for the encoder are shown in Figures 3.a and 3.b, respectively.

Since we have a strict equivalence between both canonical forms H and G in the classical single input case, i.e. r=1 and g_{10}=1, the optimization of the turbo-codes can be performed either using the canonical

form G or H. However, we show next that it is not true in the case of multiple inputs.

B. Multiple input case (r>1):

We start with the following remark. Let G and H denote the sets of the encoders with canonical forms G and H, respectively. The sizes of the matrices H and G defined in (18) and (19) are $m\times(r+2)$ and $(m+1)\times(r+1)$, respectively. Assuming $g_{0,0}=1$ and $m\geq r$, the matrix H has more entries than G. Therefore the set H is larger than the subset G, i.e. the canonical forms H and G are generally not one-to-one equivalent in the multiple input case.

The conditions for which both canonical forms are equivalent are summarized in the next two theorems.

Theorem 2: For any generator matrix H of canonical form H, it exists a unique equivalent matrix G of canonical form G, which is solution of the following system:

(α) $g_{0,0}=1$,

(β) $g_{i,0}=h_{i,1}$ for any $i=1, \ldots, r$,

(γ) $g_{0,i}=h_{0,i}$ for any $i=1, \ldots m$,

(δ) $D\cdot P_m(D)\cdot G_0=(h_{r+1}(D)\cdot H_R+h_0(D)\cdot H_E)\cdot\Delta_m(D)\cdot H_0+$
$+(h_0(D)+1)\cdot\cdot[h_{r,1}\ \ldots\ h_{2,1}\ \ h_{1,1}]$.

Proof: Equation (α) imposes the structure to be recursive. After some basic manipulations, it can easily be shown that (β), (γ) and (δ) are equivalent to (9), (16) and (17). In order to complete the proof, we have to show that (δ) has a unique solution G_0.

Lemma 1: Equation (δ) is equivalent to the equation: $G_T = A\cdot H_0$ where the $m\times m$ matrix A is determined recursively as follows:

- Initialization: $H_{ER} = [H_E\ 1]^{\mathrm{T}}\cdot H_R + [H_R\ 1]^{\mathrm{T}}\cdot H_E$ and $A(1,:) = H_{ER}(1,:)$;
- Iteration $i=2,3,\ldots m$: $A(i,:)=H_{ER}(i,:)+A(i\text{-}1,:)\cdot I_m(2)$;

where $I_m(p)$ is the $m\times m$ identity matrix shifted by p-1 positions to the right. $M(i,:)$ denotes the row -vector of the entries of the i-th row in the matrix M (Matlab notations).

Proof: We have by definition: $h_{r+1}(D)\cdot H_R+h_0(D)\cdot H_E = [D^m\ D^{m-1}\ \ldots\ D\ 1]\cdot\cdot([H_E\ 1]^T\cdot H_R+[H_R\ 1]^T\cdot H_E)$, and $\Delta_m(D) = I_m(1)\cdot D + I_m(2)\cdot D^2 + \ldots + I_m(m)\cdot D^m$. Using ($\beta$), ($\gamma$) and (12), we can now write:

$$\sum_{i=1}^{m} D^i\cdot G_T(i,:) = \sum_{i=1}^{m+1}\sum_{p=1}^{m} D^{i+p-1}\cdot H_{ER}(i,:)\cdot I_m(p)\cdot H_0 \qquad (20)$$

or equivalently term by term:

$$G_T(i,:) = \sum_{k=1}^{i} H_{ER}(i,:)\cdot I_m(i-k+1)\cdot H_0 =$$

$$= \left[H_{ER}(i,:) + \sum_{k=1}^{i-1} H_{ER}(i.:)\cdot I_m(i-k)\cdot I_m(2)\right]\cdot H_0 \ . \qquad (21)$$

We define A as: $A(i,:) = \sum_{k=1}^{i} H_{ER}(i,:)\cdot I_m(i-k+1)$. Using (21), the i-th row of A can be expressed as: $A(i,:) = H_{ER}(i,:) + A(i\text{-}1,:)\cdot I_m(2)$ and $G_T(i,:)$ is equal to $A(i,:)\cdot H_0$, $i = 1, 2, \ldots m$ or equivalently: $G_T = A\cdot H_0$.

According to the Lemma 1, for any encoder defined by its generator matrix H_0, it exists a unique generator matrix G_T and therefore a unique equivalent encoder with a canonical form G which completes the proof of Theorem 2.

Theorem 2 shows that for any matrix H there is an equivalent matrix G. We propose next to investigate if the converse is true. We start with the following example: Assume $m=r=1$ and a generator polynomial $G = [2\ 3]$. The corresponding encoder is depicted in Fig.4. This example does not belong to the classical binary case because $g_{10}=0\neq 1$. We have successively: $G_R=[1]=H_R$, $G_L=[0]=[h_{11}]=H_0$, $H=[H_E^T\ \ H_0\ \ H_R^T] = [x\ 0\ 1]$. It is easy to show that the condition (δ) is $[1]=[1+x]\cdot[0]$ which cannot be satisfied for at least one value of $x\in\{0, 1\}$. Thus, the encoder presented in Fig.4 cannot be represented in the canonical form H.

Theorem 3: For any generator matrix in G, it exists i) none, ii) one or iii) several equivalent generator matrices in H.

Proof: A generator matrix H in H which has an equivalent in G verifies the system of equations: (β) $[h_{r,1}\ \ h_{r-1,1}\ \ \ldots\ \ h_{1,1}]=G_L$; ($\gamma$) $H_R=G_R$; (δ) $A\cdot H_0=G_T$ (Lemma 1). In order to find the vector H_E and the matrix H_0 that verified (δ), a exhaustive computational search is performed through the $2^{(m-1)\cdot r+m}$ possible pairs $\{H_E, H_0\}$. Equation (δ) has i) none, ii) one or iii) several solutions:

i) We have shown in the previous example that an encoder in G may not have an equivalent in H. In section 5.A, we provide several examples.

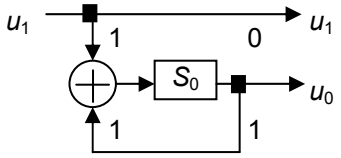


Fig. 4. Encoder for G=[2 3]

ii) There is a unique solution for the single input classical binary case as shown in Theorem 2. Moreover, for any $r>1$, one could find a unique solution H_0 to (δ) for particular values of G_T.

iii) Clearly we have: $|H|\geq|G|$, where $|.|$ denotes the cardinal number of a set. However, from Theorem 2, we know that any generator matrix in H has a unique equivalent in G. Thus, there is at least two encoders in H with the same equivalent encoder in G.

According to Theorem 3, the system of equations (α), (β), (γ) and (δ) establishes a function of equivalence ξ: H $\rightarrow$G, which is neither injective nor subjective.

Theorem 3 has a huge impact for the design of TC. Indeed, searching a good encoder in G instead of H is not only faster according to the condition (iii) but can furthermore lead to a better solution which does not exist in H according to the condition (i). Furthermore, we show in the following theorem that there are no distinct encoders in G that are equivalent to each other.

Theorem 4: $\forall \{G^1, G^2 \neq G^1\} \in G$, G^1 and G^2 are not equivalent i.e. two distinct matrices in G cannot be equivalent.

Proof: Assume that two matrices G^1 and G^2 in G are equivalent. According to (16) and (17), it implies that: $g_0^1(D) = g_0^2(D)$, $G_R^1 = G_R^2$ and $D \cdot P_m(D) \cdot G_0^1 + G_L^1 = D \cdot P_m(D) \cdot G_0^2 + G_L^2$. The last equality is verified if only if: $G_0^1 \equiv G_0^2$ and $G_L^1 \equiv G_L^2$ so $G^1 = G^2$.

C. Extension to the Multi-Non-Binary Convolutional Codes:

It is worth noting that all previous calculations are valid for binary and non-binary inputs. Therefore, all the results can be generalized for (u_i, s_j, h_{ij}, g_{ij}) in $GF(2^q)$, q>1. They can directly be used for the design of the MNBTC [9]. In that case, each connection corresponds to a q bits-wide bus; the coefficients h_{ij} and g_{ij} correspond to a multiplication circuit over $GF(2^q)$; the shift registers buffer packets of q bits, and the adders perform modulo 2 symbol-wise additions.

V. SIMULATION RESULTS

In this section, we numerically evaluate the importance of designing the MBTC by searching the candidates among the encoders with canonical form G instead of the candidates among the encoders with canonical form H:

TABLE I
CARDINAL NUMBER OF THE SETS G_o, H_o AND $\xi(H_o)$ FOR THE CODES OF RATE-$r/(r+1)$ WITH MEMORY m=2,3,4 AND WITH r INPUTS, r=1,2,3.

m	r	$\lvert G_o \rvert$	$\lvert H_o \rvert$	$\lvert \xi(H_o) \rvert$
2	1	16	21	15
	2	36	18	18
3	1	80	233	78
	2	552	663	411
	3	1568	784	756
4	1	352	2169	348
	2	5712	15189	4887
	3	48832	56588	32704

- From Section 4.3, we have $|H| \geq |G|$. Therefore, the search for good M(N)BTC in G instead of H is faster. However, for practical values of m and r, i.e. for constituent codes with memory 3 or 4, what is the effective gain?
- In Theorem 3 we showed that we could find codes in G that do not have any equivalent in H. However, for practical values of m and r, are these codes among the best constituent codes for MBTC?

A. Cardinal number of the subsets G, H and $\xi(H)$

Without loss of generality, we limit our search to the recursive decodable codes in G and H, i.e. for G_R and H_R with at least one non zero entry. Moreover, switching the inputs gives an equivalent code. Therefore we assume that the first r elements of the encoder matrix G are sorted in decreasing order. The two resulting subsets are G_o and H_o. In addition, we define the subset $\xi(H_o)$ of G_o whose each element has at least one equivalent in H_o. In Table 1, we compare the cardinal number of the sets G_o, H_o and $\xi(H_o)$ for practical values of m and r. The difference $|G_o| - |\xi(H_o)|$ exponentially increases with respect to r. The ratio $|H_o|/|\xi(H_o)|$ exponentially increases with respect to m-r. The larger the memory and/or the lower the coding rate, the more interesting the search among the encoders with canonical form G instead of H.

If the feedback polynomial $g_0(D)=h_0(D)$ is a primitive polynomial with $g_{0,m}=h_{0,m}\neq 0$, which is practically the case of all `good' codes, any matrix in G has 2^{m-r} equivalent matrices in H. Therefore, even if the search is limited to the encoding matrices with a primitive feedback polynomial, the gain of searching the constituent codes of the MBTC in G instead of H is of the same order, i.e. the computational complexity of an exhaustive search decreases exponentially with respect to m-r. To illustrate this claim, we determine in Table II the equivalent matrices for two rate-2/3 codes from [8] of memory m=3 and m=4 with generator matrices H=[6 7 1 5] and H=[11 11 1 12], respectively.

B. Optimization Results

In this section, we present the optimization results of an exhaustive search among all encoders in G, i.e., all encoders that can be represented with observer canonical form.

In order to reduce the computational complexity of the MBTC based on an exhaustive search, we perform the exhaustive search in both following steps: in the first step, we select both codes and interleavers based on a quick estimate of the ratio between the average number of iterations to decode the MNBTC and the number of packets that were transmitted without error for a fixed number of packets. Only few packets, say roughly 1000 transmitted packets for each code, are needed to accurately estimate this ratio. Moreover, we observe in our simulations that the best code in all cases belongs to the first per cent only of the codes with fastest convergence which dramatically reduces the code optimization. In the second step, we are simulating FER at a SNR such that the FER results approximatively match with a targeted FER. In our case, we target a FER around 10^{-4} and are using 500000 codewords to estimate it: The best code is the code with the smallest FER.

For implementation considerations, we focus on double-binary turbo-codes (MBTC with two inputs) with memory m=2 and 3. In all cases except if it is notified, we use an S-interleaver of length 752 the trellis-termination technique using tail-bits to drive the encoder to the all-zero state; moreover, the coding rate is 1/2 for all MBTC.

For m=2, the best code that we found has for generator matrix G=[7 6 5] which does not have an equivalent in canonical form H.

For m>2, the codes that belong to G\H do not have a primitive feedback connection polynomial $g_0(D)$ of

maximal degree *m*. Since the best MBTC are based on a primitive feedback polynomial, it is not necessary to consider the codes that belong to G\H in the MBTC design. The canonical form G is still interesting for the MBTC design in order to simplify the exhaustive search based on Theorem 4.

For *m*=3 and an S-interleaver of parameter 22, the best MBTC that we found is the code proposed in [8] with generator matrix G=[13 15 11]. Whereas the S-interleaver is asymptotically optimal in the codeword size [15], we also consider the family of interleaver proposed in [8] which is simpler to implement in hardware. These interleavers are defined by 4 parameters: P, P1 P2 and P3. For more details on the construction of such interleavers, we refer to [8]. Through an exhaustive search, we found two best codes with generator matrices G=[15 9 11] and G=[15 9 13] and with interleavers defined by the parameters P=19, P1=376, P2=328, P3=196 and P=19, P1=376, P2=203 and P3=677, respectively. Both codes have similar Bit Error Rate and Frame Error Rate performance and outperforms the MBTC proposed by [8]. For a frame error rate of 10e-4, the gain is approximatively equal to 0.3 decibels.

Finally, for *m*=4, the best code that we found has for generator matrix *G*=[21 23 25] which is the same that Douillard proposed in [8].

VI CONCLUSIONS

The design of the turbo-codes with multiple inputs generally requires an exhaustive search through large sets of feasible codes and interleavers. This search is a major burden in their optimization since the computational complexity of the search is exponential with the number of inputs *r*.

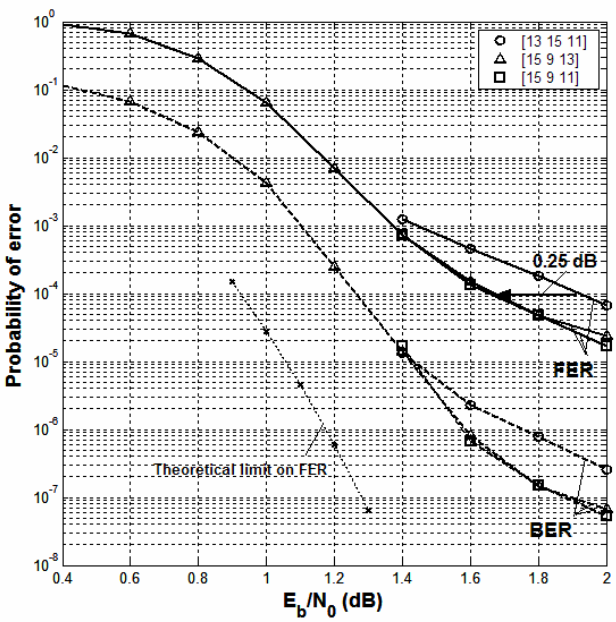


Fig.6 Bit and Frame Error Rate Performance as a function of the signal-to-noise ratio for three rate-1/2 codes with memory 3: both codes that we designed in this paper defined by generator matrices G=[15 9 13] and G=[15 9 11], respectively and the code proposed in [8] with generator matrix [13 15 11]. The theoretical limits on FER are derived from [16].

In this paper, we showed that the encoder of such codes could be represented with the observer canonical configuration. This configuration is essential for reducing the computational complexity of the search (up to 300% for the codes that we considered). Based on this strategy, we were able to design a rate-1/2 turbo-code with two inputs and memory m=3 which outperforms the equivalent turbo-code proposed by Douillard et al. by 0.25 decibels for a frame error rate of 10^{-4}.

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