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On the Asymptotic Decorrelation of the Wavelet Packet Coefficients of a Wide-Sense Stationary Random Process

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***Abstract*— Consider the wavelet packet coefficients issued from the decomposition of a random process stationary in the wide-sense. We address the asymptotic behaviour of the autocorrelation of these wavelet packet coefficients. In a first step, we explain why this analysis is more intricate than that already achieved by several authors in the case of the standard discrete orthonormal wavelet decomposition. In a second step, it is shown that the autocorrelation of the wavelet packet coefficients can be rendered arbitrarily small provided that both the decomposition level and the regularity of the quadrature mirror filters are large enough.**

I. Introduction

CONSIDER a second-order random process and assume that this random process is stationary in the wide-sense. The discrete orthonormal wavelet and the wavelet packet decompositions of this process yield coefficients that are random variables. Many authors have studied the statistical correlation of these coefficients, see [1]–[6] amongst others. In the case of discrete orthonormal wavelet transform, in-scale coefficients tend to be uncorrelated when the decomposition level increases. At first sight, it would seem quite reasonable to consider that the same property remains valid for wavelet packet coefficients. Unfortunately, the analysis of the autocorrelation of these wavelet packet coefficients is significantly more intricate than expected, mainly because of the role played by the regularity of the quadrature mirror filters. This analysis is presented below. The proofs of the several theoretical results stated hereafter are postponed to a forthcoming paper because of the limited size of the present one.

This paper in organized as follows. In section II, the reader is reminded with basic results concerning the wavelet packet decomposition of a wide-sense stationary random process. In particular, for a given decomposition level, we give the expression of the autocorrelation function of the discrete sequence formed by the wavelet packet coefficients. The asymptotic behaviour of this function is then achieved in two steps. In section III, the analysis is worked out in the case of the ideal Shannon wavelet packet decomposition, which employs ideal quadrature mirror filters [7]. Since quadrature mirror filters such as the Daubechies and Battle-Lemarié filters tend to ideal filters when their regularity increases, the asymptotic behaviour of the autocorrelation function of the wavelet packet coefficients when such filters are used derives from that described in section III. This asymptotic behaviour is stated in section IV. It depends on the wavelet packet decomposition level as well as the regularity of the filters at hand.

II. The wavelet packet decomposition of a wide-sense stationary random process

A. Wavelet packet decomposition

Let m_0 and m_1 be the Fourier transform of two quadrature mirror filters such that

$$m_0(\omega) = \frac{1}{\sqrt{2}} \sum_{\ell \in \mathbb{Z}} h_0[\ell] e^{-i\ell\omega}, \tag{1}$$

and

$$m_1(\omega) = \frac{1}{\sqrt{2}} \sum_{\ell \in \mathbb{Z}} h_1[\ell] e^{-i\ell\omega}. \tag{2}$$

Let Φ be the scaling function associated to m_0. We define the sequence $(W_n)_{n \geq 0}$ of elements of $L^2(\mathbb{R})$ by recursively setting

$$W_{2n}(t) = \sqrt{2} \sum_{\ell \in \mathbb{Z}} h_0[\ell] W_n(2t - \ell) \tag{3}$$

and

$$W_{2n+1}(t) = \sqrt{2} \sum_{\ell \in \mathbb{Z}} h_1[\ell] W_n(2t - \ell) \tag{4}$$

with $W_0 = \Phi$. We then have $W_1 = \Psi$, where Ψ is the wavelet function associated to the quadrature mirror filters under consideration. If we now put

$$W_{j,n}(t) = 2^{-j/2} W_n(2^{-j} t), \tag{5}$$

and

$$W_{j,n,k}(t) = \tau_{2^j k} W_{j,n}(t) = 2^{-j/2} W_n(2^{-j} t - k), \tag{6}$$

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the set $\{W_{j,n,k} : k \in \mathbb{Z}\}$ of *wavelet packets* is an orthonormal system of vectors of the Hilbert space $L^2(\mathbb{R})$. With a slight abuse of language, the vector space $\mathbf{W}_{j,n}$ generated by $\{W_{j,n,k} : k \in \mathbb{Z}\}$ will hereafter be called the *packet* $\mathbf{W}_{j,n}$. For every $j = 0, 1, 2, \cdots$, and every $n \in I_j = \{0, 1, \cdots, 2^j - 1\}$, the wavelet packet decomposition of the function space $\mathbf{W}_{0,0}$ is obtained by recursively applying the so-called *splitting lemma* [8] to every space $\mathbf{W}_{j,n}$. We thus can write that

$$\mathbf{W}_{j,n} = \mathbf{W}_{j+1,2n} \oplus \mathbf{W}_{j+1,2n+1}. \quad (7)$$

The sets $\{W_{j+1,2n,k} : k \in \mathbb{Z}\}$ and $\{W_{j+1,2n+1,k} : k \in \mathbb{Z}\}$ are orthonormal bases of the vector spaces $\mathbf{W}_{j+1,2n}$ and $\mathbf{W}_{j+1,2n+1}$, respectively. The decomposition tree of figure 1 illustrates such a decomposition

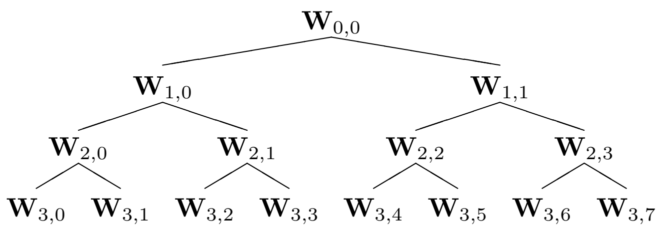

Fig. 1. Wavelet packet decomposition tree down to decomposition level $j = 3$.

Remark 1: given $j \in \mathbb{N}$, consider a binary sequence $(\epsilon_\ell)_{\ell\in\{1,2,\ldots,j\}}$ of $\{0,1\}^j$. Basically, this sequence corresponds to the sequence $m_{\epsilon_1}, m_{\epsilon_2}, \ldots, m_{\epsilon_j}$ of filters successively applied to calculate the coefficients of the packet $\mathbf{W}_{j,n}$ where

$$n = \sum_{\ell=1}^{j} \epsilon_\ell 2^{j-\ell}. \quad (8)$$

Readily, n is an element of I_j and the sequence $(\epsilon_\ell)_{\ell\in\{1,2,\ldots,j\}}$ is the unique path issued from $\mathbf{W}_{0,0}$ that leads to $\mathbf{W}_{j,n}$ in the wavelet packet decomposition tree. Conversely, let $n \in I_j$. There exists a unique sequence $(\epsilon_\ell)_{\ell\in\{1,2,\ldots,j\}}$ of $\{0,1\}^j$ such that equation (8) holds true. In the sequel, when a natural number n and a binary sequence $(\epsilon_\ell)_{\ell\in\{1,2,\ldots,j\}}$ of $\{0,1\}^j$ satisfy (8), we will say that n and $(\epsilon_\ell)_{\ell\in\{1,2,\ldots,j\}}$ are associated to each other.

Proposition 1: Consider a function W_n (defined from the recurrence (3) and (4)) where n has the form (8) for some $j > 0$. The Fourier transform $\hat{W}_n$ of W_n is given, for every real value ω, by

$$\hat{W}_n(\omega) = \left[\prod_{\ell=1}^{j} m_{\epsilon_\ell}(\frac{\omega}{2^{j+1-\ell}})\right] \hat{W}_0(\frac{\omega}{2^j}), \quad (9)$$

where $(\epsilon_\ell)_{\ell\in\{1,2,\ldots,j\}}$ is the binary sequence associated to n.

B. The autocorrelation function of the wavelet packet coefficients of a wide-sense stationary random process

Let $X : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ be a second-order, centred and wide-sense stationary random process where Ω is some probability space. The autocorrelation function of this random process is denoted by $R_X(t,s) = \mathbb{E}[X(t)X(s)] = R_X(t-s)$. We assume that X is continuous in quadratic mean. Then, R_X is a continuous function. We also assume that X has a power spectral density γ_X, which is the Fourier transform of R_X.

Given a decomposition level j and $n \in I_j$, the wavelet packet decomposition of X returns, at node (j,n), the random variables

$$c_{j,n}[k] = \int_{\mathbb{R}} X(t) W_{j,n,k}(t) dt, \quad k \in \mathbb{Z}, \quad (10)$$

provided that the integral

$$\iint_{\mathbb{R}^2} R_X(t,s) W_{j,n,k}(t) W_{j,n,k}(s) dt ds$$

exists [2].

Let $R_{c_{j,n}}$ stand for the autocorrelation function of the discrete processus $c_{j,n}$ defined by (10). It can be shown that, for every $m \in \mathbb{Z}$

$$R_{c_{j,n}}[m] = \frac{1}{2\pi} \int_{\mathbb{R}} \gamma_X(\frac{\omega}{2^j}) |\hat{W}_n(\omega)|^2 e^{im\omega} d\omega. \quad (11)$$

Our purpose is then to analyse the behaviour of this function for large values of j. Since $n \in I_j$, the analysis must take this dependence into account. If n is constant with j, Lebesgue's dominated convergence theorem can be used to compute the limit of $R_{c_{j,n}}[m]$ when j grows to infinity. If $n = 0$, the result thus obtained is that given in [4], [5]. The situation becomes more intricate if n is a function of j. For instance, if we choose $n = 2^{j-L}$ where $L \in \{1, \ldots, j-1\}$, the behaviour of $R_{c_{j,n}}[m]$ when j grows to infinity is no longer a straightforward consequence of Lebesgue's dominated convergence theorem.

The approach proposed below embraces these several cases by considering the binary sequence associated to a node (j,n) of the decomposition tree. By so proceeding, the crucial role played by the regularity of the quadrature mirror filters is enhanced.

III. Asymptotic behaviour of the wavelet packet coefficients for the Shannon wavelet packet decomposition

The Shannon wavelet packet decomposition corresponds to the case where the scaling function Φ is $\Phi^S = sinc$. The quadrature mirror filters of this decomposition are the ideal low and high pass filters $m_0^S(\omega) = \sqrt{2} \sum_{\ell\in\mathbb{Z}} \chi_{\Delta_0}(\omega - 2\pi\ell)$ and $m_1^S(\omega) = \sqrt{2} \sum_{\ell\in\mathbb{Z}} \chi_{\Delta_1}(\omega - 2\pi\ell)$, where χ_Δ stands for the indicator function of the set Δ, $\Delta_0 = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and $\Delta_1 = \left[-\pi, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right]$. The Fourier transform $\hat{W}_0^S$ of the scaling function is then $\hat{W}_0^S = \hat{\Phi}^S = \chi_{[-\pi,\pi]}$.

According to Coifman et Wickerhauser ([9], [7, pp. 326-327]), for every $j > 0$ and every $n \in I_j$, there exists a unique $p = G[n] \in I_j$ such that $|\hat{W}_{j,n}^S(\omega)| = 2^{j/2} \chi_{\Delta_{j,p}}(\omega)$, where W_n^S stands for the map W_n recursively defined by (3, 4) when the pair of quadrature mirror filters is (m_0^S, m_1^S) and

$$\Delta_{j,p} = \left[-\frac{(p+1)\pi}{2^j}, -\frac{p\pi}{2^j}\right] \cup \left[\frac{p\pi}{2^j}, \frac{(p+1)\pi}{2^j}\right]. \quad (12)$$

The map G permutes the elements of I_j and we can prove that

$$G[2n+\epsilon] = 3G[n] + \epsilon - 2\left\lfloor \frac{G[n]+\epsilon}{2} \right\rfloor, \quad (13)$$

where $\epsilon \in \{0,1\}$ and $\lfloor z \rfloor$ is the largest integer less than or equal to z.

With the same notations as those introduced above, we define, for every natural number j,

$$\gamma_j(\omega) = \sum_{\ell=0}^{2^j-1} \gamma_X(\frac{\ell\pi}{2^j})\ \chi_{\Delta_{j,\ell}}(\omega), \tag{14}$$

where $\Delta_{j,\ell}$ is defined according to (12). We then have the following result.

Proposition 2: Let j be some natural number and n be any element of I_j. We have that

$$\frac{1}{2\pi}\int_{\mathbb{R}} \gamma_j(\frac{\omega}{2^j})|\hat{W}_n^S(\omega)|^2 e^{im\omega} d\omega = \gamma_X(\frac{p\pi}{2^j})\delta[m], \tag{15}$$

where $\delta[m] = 1$ if $m = 0$ and $\delta[m] = 0$ otherwise, $p = G[n]$ and G is given by (13).

In what follows, given an arbitrary infinite binary sequence $\kappa = (\epsilon_k)_k \in \{0,1\}^{\mathbb{N}}$ and any integer j, n_j will stand for the natural number associated to the finite subsequence $(\epsilon_k)_{k=1,2,\ldots,j}$ and we set $p_j = G[n_j]$.

It is easy to see that the sequences $(\frac{n_j\pi}{2^j})_j$ and $(\frac{p_j\pi}{2^j})_j$ are Cauchy. As such, each of them has a unique limit. In particular, the limit

$$a(\kappa) = \lim_{j\to+\infty} \frac{p_j\pi}{2^j}, \tag{16}$$

will play a crucial role in the sequel. Table I displays the value of $a(\kappa)$ for the sequences κ that were employed to carry out the experiments whose results are given in section V. These sequences are $\kappa_1 = (0,0,0,0,0,\cdots), \kappa_2 = (0,0,1,0,0,\cdots), \kappa_3 = (0,1,0,0,0,\cdots)$, and $\kappa_4 = (1,0,0,0,0,\cdots)$. The sequences $(n_j)_j$ and $(p_j)_j$ corresponding to these sequences are given in table I.

TABLE I

VALUE OF $a(\kappa) = \lim_{j\to+\infty} \frac{p_j\pi}{2^j}$ WITH RESPECT TO $(n_j)_j$

Sequence	κ_1	κ_2	κ_3	κ_4
n_j for $j \geq 3$	0	2^{j-3}	2^{j-2}	2^{j-1}
p_j for $j \geq 3$	0	$2^{j-2}-1$	$2^{j-1}-1$	2^j-1
$a(\kappa)$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π

Theorem 1: Let $\kappa = (\epsilon_k)_{k\in\mathbb{N}}$ be a binary sequence of $\{0,1\}^{\mathbb{N}}$. With the notations introduced just above, consider the packets $\mathbf{W}^S_{j,n_j}$, $j \in \mathbb{N}$.

Let $R^S_{c_{j,n_j}}$ be the autocorrelation function of the Shannon wavelet packet coefficients c_{j,n_j} at node (j,n_j). If $a(\kappa)$ is a continuity point of γ_X, then

$$\lim_{j\to+\infty} R^S_{c_{j,n_j}}[m] = \gamma_X(a(\kappa))\delta[m], \tag{17}$$

Remark 2: the autocorrelation function $R^S_{c_{j,n_j}}$ of the Shannon wavelet packet coefficients c_{j,n_j} at node (j,n_j) derives from (11) and is thus given by

$$R^S_{c_{j,n_j}}[m] = \frac{1}{2\pi}\int_{\mathbb{R}} \gamma_X(\frac{\omega}{2^j})|\hat{W}^S_{n_j}(\omega)|^2 e^{im\omega} d\omega. \tag{18}$$

IV. ASYMPTOTIC BEHAVIOUR OF THE AUTOCORRELATION FUNCTION OF THE WAVELET PACKETS ASSOCIATED TO A WIDE-SENSE STATIONARY PROCESS

We now consider non-ideal quadrature mirror filters m_0 et m_1. It is known that the multiplicity of the zero of m_0 in π equals the number of null moments of the analysing wavelet when m_0 is either a Daubechies or a Battle-Lemarié filter. Furthermore, the wavelet regularity increases with the number of its null moments. We then say that the regularity of a pair (m_0, m_1) of quadrature mirror filters is r if the scaling filter can be written in the form

$$m_0(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right)^r Q(e^{-i\omega}). \tag{19}$$

The scaling filter has thus a zero with multiplicity r in $\omega = \pi$. This notion of regularity relates to the flatness of the filter magnitude response.

According to [10]–[12], the standard Daubechies filters converge pointwise to the ideal Shannon filters when their regularity tends to infinity. Figure 2 illustrates this convergence by displaying the magnitude response of the Daubechies scaling filters with regularity $1, 2, 4, 10, 20$, and 40.

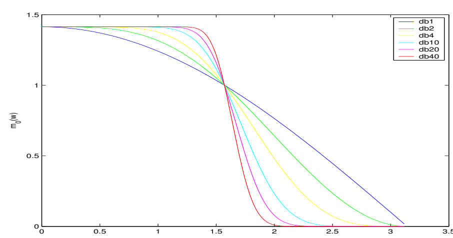

Fig. 2. Magnitude response of Daubechies scaling filters. In this figure, dbr stands for the rth-order Daubechies scaling filter.

The Battle-Lemarié filters satisfy the same property [13].

In what follows, we consider rth-order quadrature mirror filters that are denoted by $(m_\epsilon^{[r]})_{\epsilon\in\{0,1\}}$. We then can state the subsequent result where the notations introduced so far are used with the same meaning as above.

Theorem 2: Let X be a second-order random process. Assume that X is centred, stationary in the wide-sense and continuous in quadratic mean.

Assume that the power spectral density γ_X of X is bounded, with support in $[-\pi,\pi]$ and continuous at $a(\kappa)$ where κ is some binary sequence of $\{0,1\}^{\mathbb{N}}$.

For every given regularity r, the wavelet packet coefficients of $\mathbf{W}^{[r]}_{j,n_j}$ form a second-order discrete random process whose correlation function derives from (11) and is equal to

$$R^{[r]}_{c_{j,n_j}}[m] = \frac{1}{2\pi}\int_{\mathbb{R}} \gamma_X(\frac{\omega}{2^j})|\hat{W}^{[r]}_{n_j}(\omega)|^2 e^{im\omega} d\omega. \tag{20}$$

For every given positive real number $\eta > 0$, there exists an integer j_0 with the following property : for every natural number $j \geq j_0$, there exists $r_0 = r_0(j,n_j)$ such that, for every $r \geq r_0$, $|R^{[r]}_{c_{j,n_j}}[m] - \gamma_X(a(\kappa))\delta[m]| < \eta$.

V. Experimental results

A. The role played by the decomposition level

We consider a wavelet packet decomposition tree whose depth is $J = 6$. We constructed a random process as follows. The wavelet coefficients of every packet at decomposition level 6 were set to centred, independent and identically Gaussien distributed random variables. The value of the variance of these random variables was randomly chosen. By using the standard wavelet packet reconstruction algorithm, we obtained the random process whose spectral density is given by figure 3. When we decompose this random process by using the same quadrature mirror filters as those used for synthesizing it and consider the packets $\mathbf{W}_{j,n_j}$ when n_j is associated to the sequences $\kappa_1, \kappa_2, \kappa_3$, and κ_4 of table I, we note that, for $j = 3$, some coeffients c_{j,n_j} remain strongly correlated whereas some others are not (see figure 5).

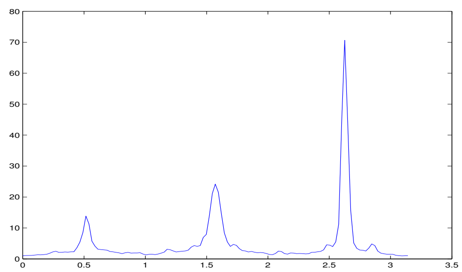

Fig. 3. Spectral density of the random process used to illustrate the role played by the decomposition level.

B. Influence of the regularity

We decompose the same random process as that used above and whose spectral density is displayed in figure 3. Now, we use the Daubechies filters with regularity 1 and 20. For the third decomposition level ($j = 3$), the results thus obtained are those of figure 6. This illustrates the role played by the regularity of the quadrature mirror filters.

C. The limit value of the correlation function

According to theorem 2, if the decomposition level and the regularity of the filters are both large enough, the correlation functions must tend to $\gamma_X(a(\kappa))\delta[m]$ where $\gamma_X(a(\kappa))$ is the value of the spectral density of the random process at $a(\kappa)$, $a(\kappa)$ is given by (16) and κ is some binary sequence.

Let us consider the random process whose spectral density is that of figure 4.

Quite rapidly, the value of the autocorrelation function at the origin becomes close to $\gamma_X(a(\kappa))$. This is pointed out by figure 7. This figure displays the autocorrelation functions obtained at the sixth level of the wavelet packet decomposition tree for the wavelet packets respectively associated to the sequences $\kappa_1, \kappa_2, \kappa_3$, and κ_4 of table I when the quadrature mirror filters are the Daubechies filters with regularity 1, 4 and 10. The same figure also pinpoints that all the wavelet packet coefficients become reasonably uncorrelated when the regularity of the filters is large enough. This illustrates also and once again the crucial role played by the regularity of the quadrature mirror filters.

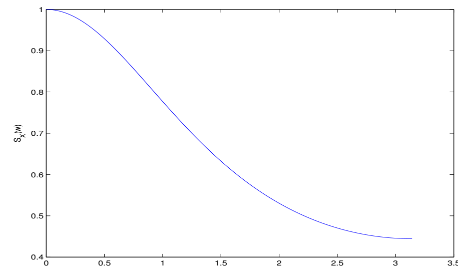

Fig. 4. Spectral density function of the random process employed to compute the limit value of the correlation function. This random process was synthesized by filtering some white noise with an autoregressive filter.

VI. Conclusion

This paper provides further details concerning the asymptotic behaviour of the autocorrelation function of the wavelet packet coefficients issued from the decomposition of a wide-sense stationary random process. By choosing a sufficiently large decomposition level and, then, by increasing the regularity of the filters with respect to the chosen decomposition level, the wavelet packet coefficients tend to become uncorrelated.

The result presented in this paper complements thos established in [1]–[5] and justifies the assumption on which many signal processing techniques based on wavelet packet decompositions are based, namely, that signals are corrupted by white noise.

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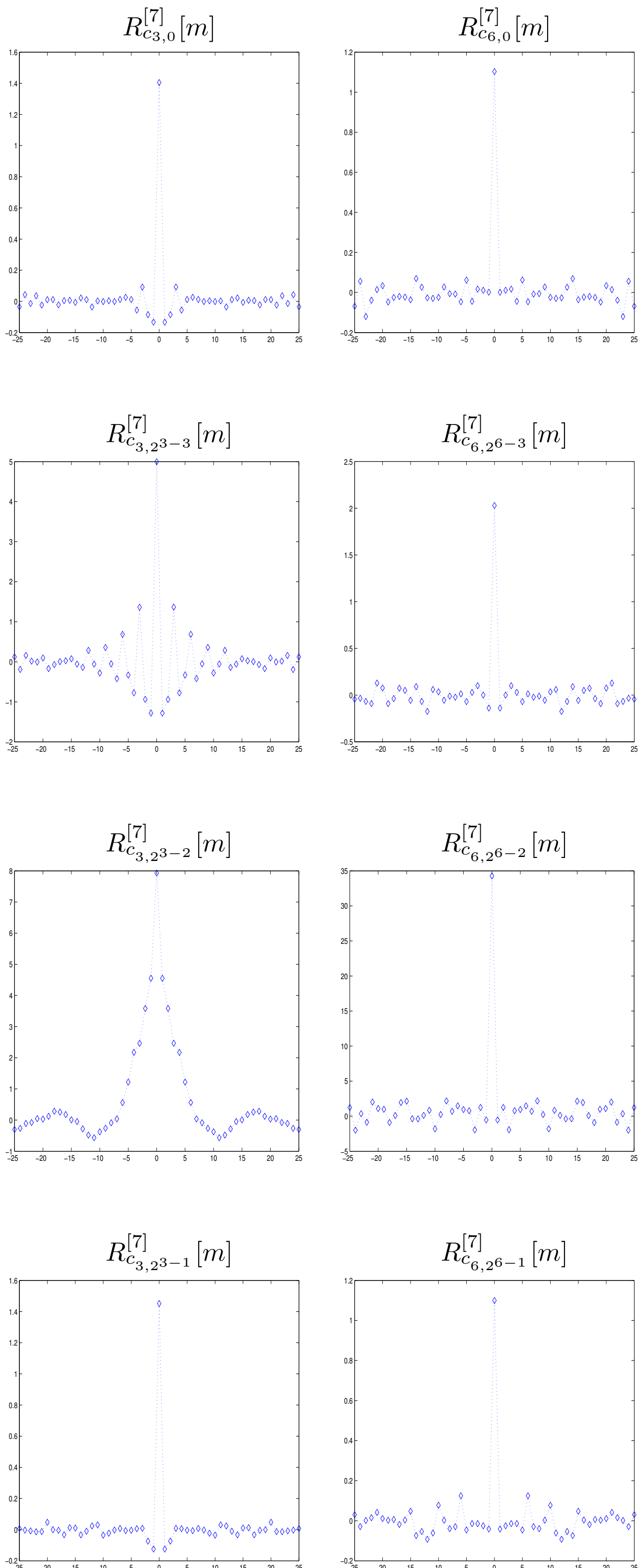

Fig. 5. Autocorrelation function of the wavelet packet coefficients returned by the decomposition of the random process whose spectral density is that of figure 3. The decomposition was achieved by using the Daubechies filters with regularity 7. The first column displays results obtained for $j = 3$ whereas the second column concerns $j = 6$ where j stands for the decomposition level.

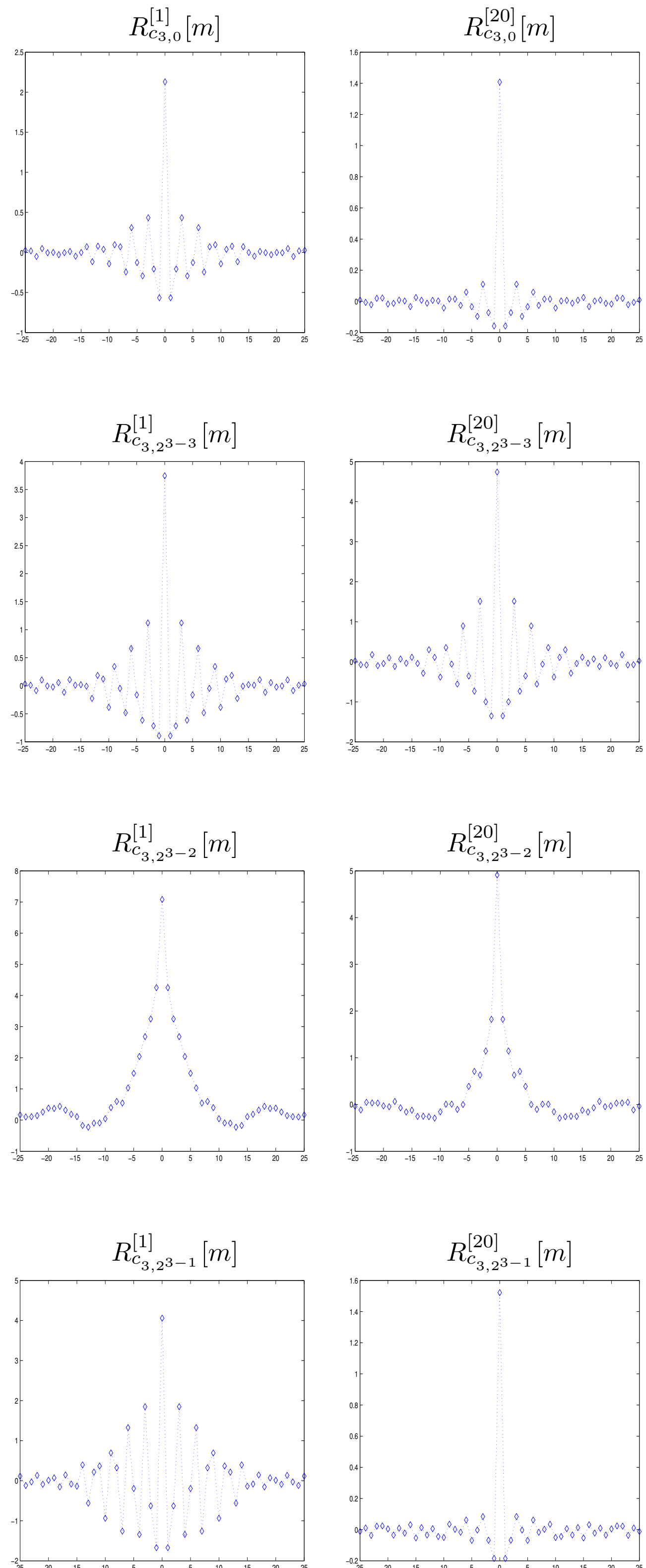

Fig. 6. Autocorrelation functions of the wavelet packet coefficients at decomposition level $j = 3$ for the random process with power spectral density given by figure 3. The first column displays the results obtained with the Daubechies filters of regularity 1; the second column presents the results obtained by using the Daubechies filters with regularity 20.

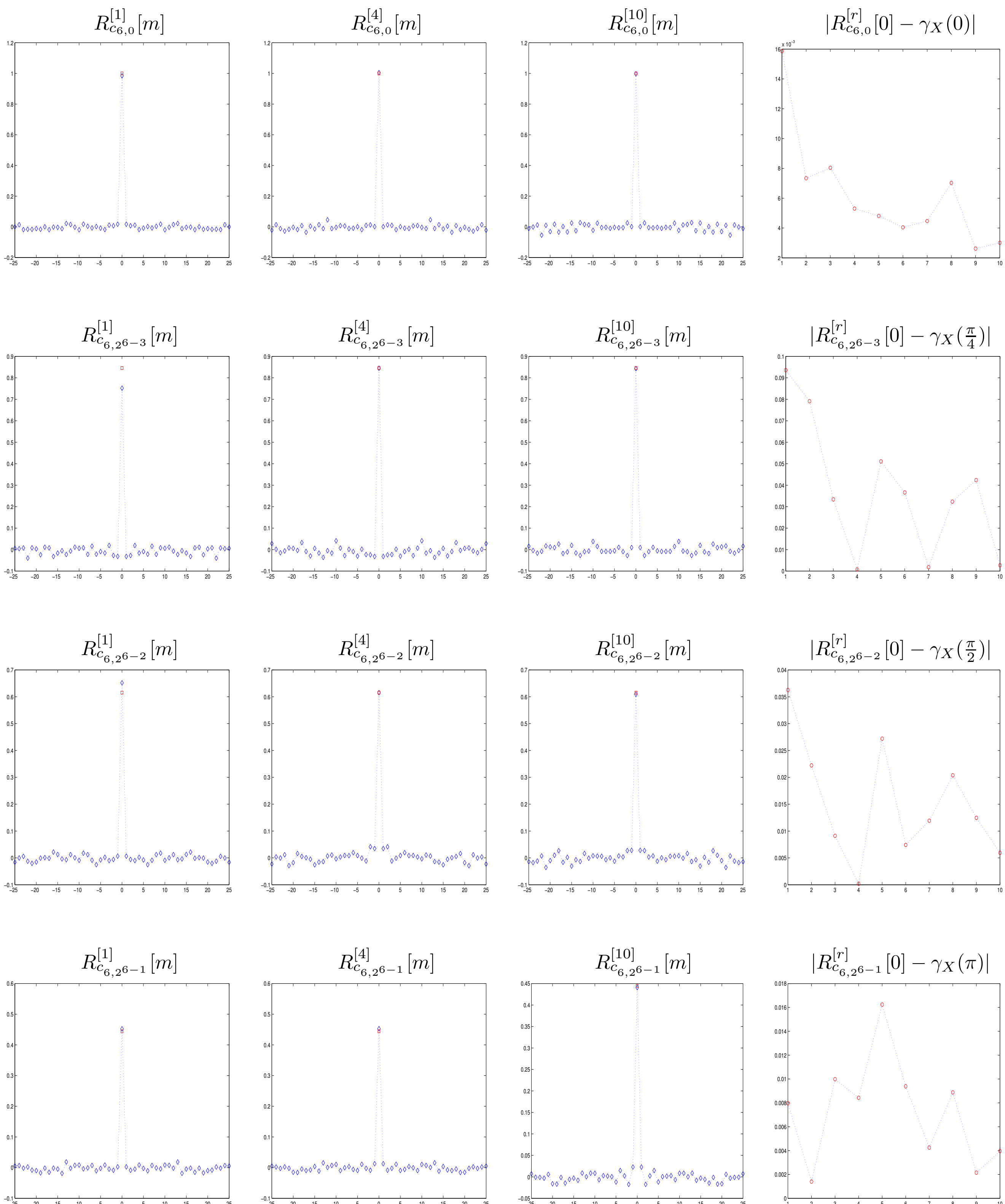

Fig. 7. The first third columns display the autocorrelation functions of the wavelet packet coefficients obtained by decomposing the random process whose spectral density is given by figure 4. The values of these functions are representated by $-\diamond$'s. On each figure, the value of the spectral density function at $a(\kappa)$, where κ is a binary sequence, is represented by a $\square$. For each packet, the maximum of the autocorrelation function must be close to this square. The fourth column displays the differences $|R^{[r]}_{c_{j,n_j}}[0] - S_x(a(\kappa))|$ when the regularities of the Daubechies filters range from 1 to 10, $j = 6$ et $n_j = 0$, 2^{6-3}, 2^{6-2}, et 2^{6-1}.