# An Improved Version of the Inverse Hyperanalytic Wavelet Transform

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*Abstract* — **The success of wavelet techniques in many fields of signal and image processing was proved to be highly influenced by the properties of the wavelet transform used, mainly the shift-invariance and the directional selectivity. In the present paper we propose an improved version of the inverse Hyperanalytic Wavelet Transform (HWT), which uses hyperanalytic mother wavelets. We have already proposed implementations of the HWT and of its inverse (IHWT). The implementation supposes the computation of the discrete wavelet transform (DWT) of the hyperanalytic signal associated to the input signal. Our old computation method of the IHWT extracts the real part of the signal at the output of the inverse discrete wavelet transform (IDWT). The aim of this paper is a new implementation of the IHWT, which permits a better shift invariance. We will compare this implementation with our previous one, with the DWT and with Kingsbury's Double-Tree Complex Wavelet Transform (DT CWT).**

## I. Introduction

Techniques relying on the wavelet transform are widely used in various signal and image processing methods, namely in image denoising, segmentation and classification. The property of shift-invariance associated with good directional selectivity of a wavelet transform lead to improved image processing results. The first wavelet transform to assure these properties was introduced by Grossman and Morlet [1], namely the Continuous Wavelet Transform (CWT) [2], which uses continuous complex mother wavelets. CWT was not widely used in image processing due to the difficulty in designing complex filters to satisfy the perfect reconstruction property. To overcome this problem, Kingsbury introduced the DTCWT [3, 4] which is a quadrature pair of Discrete Wavelet Transform (DWT) trees, invertible and quasi shift-invariant. However the design of the quadrature wavelet pairs is rather complicated and can be done only through approximations. In [5] we have introduced the HWT, comprising advantages regarding the shift-invariance and the directional selectivity of the DTCWT, and those regarding the flexibility in choosing the mother wavelet of the DWT [6].

In the present paper we propose a new method of reconstruction of the signal from the HWT's coefficients. For simplicity reasons, we will show the implementation scheme in the one-dimensional case in section II. In section III we will present some simulation results obtained in the two-dimensional case. The paper concludes with few final remarks.

## II. The Analytical Discrete Wavelet Transform

The Analytical Discrete Wavelet Transform (ADWT) was first introduced in [7], and refers to the one-dimensional (1D) version of the HWT.

A 1D wavelet transform (WT) is shift-sensitive if a small shift in the input signal can cause major variations in the distribution of energy between DWT coefficients at different scales. The shift-sensitivity of the DWT is due to the down-samplers used in its computation.

In [2, 8] is devised the Undecimated Discrete Wavelet Transform (UDWT), which is a WT without down-samplers. Although UDWT is shift-insensitive, it has a redundancy of $2^J$, where J denotes the number of iterations of the WT and its implementation requires a large number of different filters.

Abry [9] first demonstrated that approximate shiftability is possible for the DWT with a small, fixed amount of transform redundancy. He designed a pair of real wavelets such that one is approximately the Hilbert transform of the other. This wavelet pair defines a complex wavelet transform. Kingsbury [3, 4] developed DTCWT which is a quadrature pair of DWT trees, similar to Abry's transform. Both transforms are quasi shift-invariant, but the filter design is quite complicated.

The ADWT is a complex wavelet transform, but instead of using complex mother wavelets, it uses regular mother wavelets (such those proposed by Daubechies), but the transform should be applied to the analytical signal associated to the input signal, computed as:

$$x_a = x + j\mathcal{H}\{x\} \qquad (1)$$

Our previous implementations of the ADWT and of its inverse (IADWT) are presented in fig. 1. In the implementation of the IADWT that we're proposing now, we take the mean between the real part and the Hilbert transform of the imaginary part (i.e. we invert the Hilbert transform we've applied at the entry of the transform) (fig.2).

Because the Matlab implementation of the Hilbert transform is not exactly invertible, we have implemented a new Hilbert transformer that relies on a Fast Fourier Transform (FFT) algorithm, which assures a good reversibility ($x = -\mathcal{H}\{\mathcal{H}\{x\}\}$, where $\mathcal{H}$ denotes the Hilbert transform).

In the following figure (fig.3) we have taken the test Kingsbury has done to illustrate the quasi shift-invariance of his transform [3], and we use it to make a comparison between ADWT in its new implementation, DT CWT, ADWT in its old implementation and the classical DWT. As can be seen, we obtain results comparable with those obtained by Kingsbury, even though we have used Daubechies, 10-taps filter.

The criterion we previously used to measure the invariance of a wavelet transform is the degree of invariance, defined as:

$$\mathrm{Deg} = 1 - \mathrm{d/m} \tag{2}$$

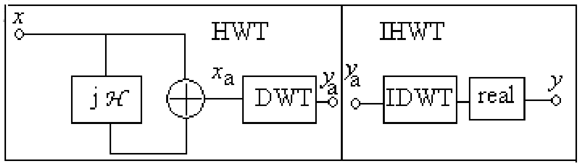


Figure 1. First implementation of the ADWT

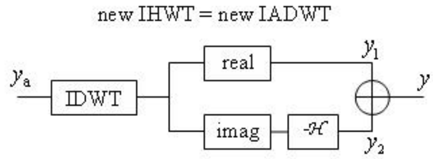


Figure 2. The new implementation of the IADWT.

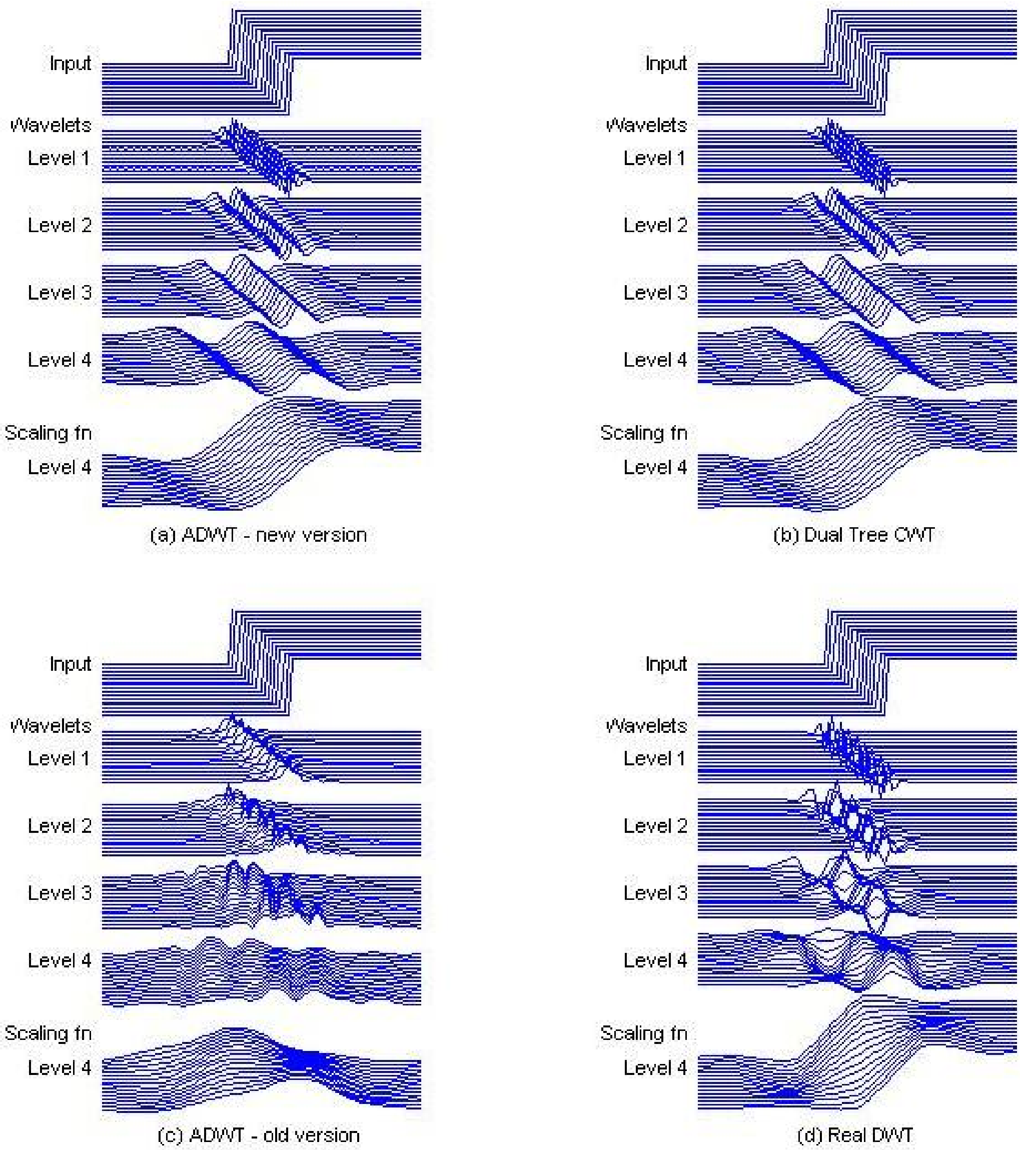


Figure 3. Comparison between ADWT, DT CWT and DWT from the shift-invariance point of view. In the case of the old variant of ADWT we have considered for reconstruction the magnitudes of the wavelet coefficients.

In this definition, we have denoted with d the standard deviation of the set of energies computed for 16 different shifts and for a particular type of coefficients (details or approximations) and m denotes the mean of those energies. The degree of shift-invariance is 1 if the transform is shift-invariant. In table 1 we have put the values of this degree computed for the signals deployed in fig. 3. As it can be seen, our new implementation is comparable with the DT CWT both from the visual point of view and by analyzing the values of the degree of invariance.

TABLE I. VALUE OF DEGREE OF INVARIANCE

| WT | Degree of invariance | | | | |
|---|---|---|---|---|---|
| | *W. lev. 1* | *W. lev. 2* | *W. lev. 3* | *W. lev. 4* | *Sn.lev. 4* |
| ADWT – new | 0.9967 | 0.9995 | 0.9986 | 0.9988 | 0.9990 |
| DT CWT | 1.0000 | 0.9811 | 0.9749 | 0.9734 | 0.9997 |
| ADWT – old | 0.9969 | 0.9982 | 0.9981 | 0.9983 | 0.9995 |
| DWT | 0.9236 | 0.8265 | 0.7878 | 0.8149 | 0.9958 |

## III. THE HYPERANALYTICAL WAVELET TRANSFORM

The Hyperanalytical Wavelet Transform refers to the 2-dimensional case of the ADWT. The definition of the HWT of an image $f(x,y)$ is:

$$\mathrm{HWT}\{f(x,y)\} = \mathrm{DWT}\{f(x,y)\} + i\mathrm{DWT}\{\mathcal{H}_x\{f(x,y)\}\} + j\mathrm{DWT}\{\mathcal{H}_y\{f(x,y)\}\} + k\mathrm{DWT}\{\mathcal{H}_y\{\mathcal{H}_x\{f(x,y)\}\}\} \quad (3)$$

From (3) we can see that the implementation of the HWT requires 4 2D-DWT transforms, applied to the original signal, to the Hilbert transform computed over the lines, to the Hilbert transform computed over the columns and to the Hilbert transform computed over the columns of the Hilbert transform computed over the lines of the original signal. In what concerns the reconstruction part, for each of the three trees mentioned above the reversed Hilbert transform is applied at the input (both Hilbert transforms in the last case), and the final result represents the mean of the four images computed on the four trees.

In fig. 4 we have taken the test Kingsbury used in [3] to illustrate the shift-invariance in the 2D case.

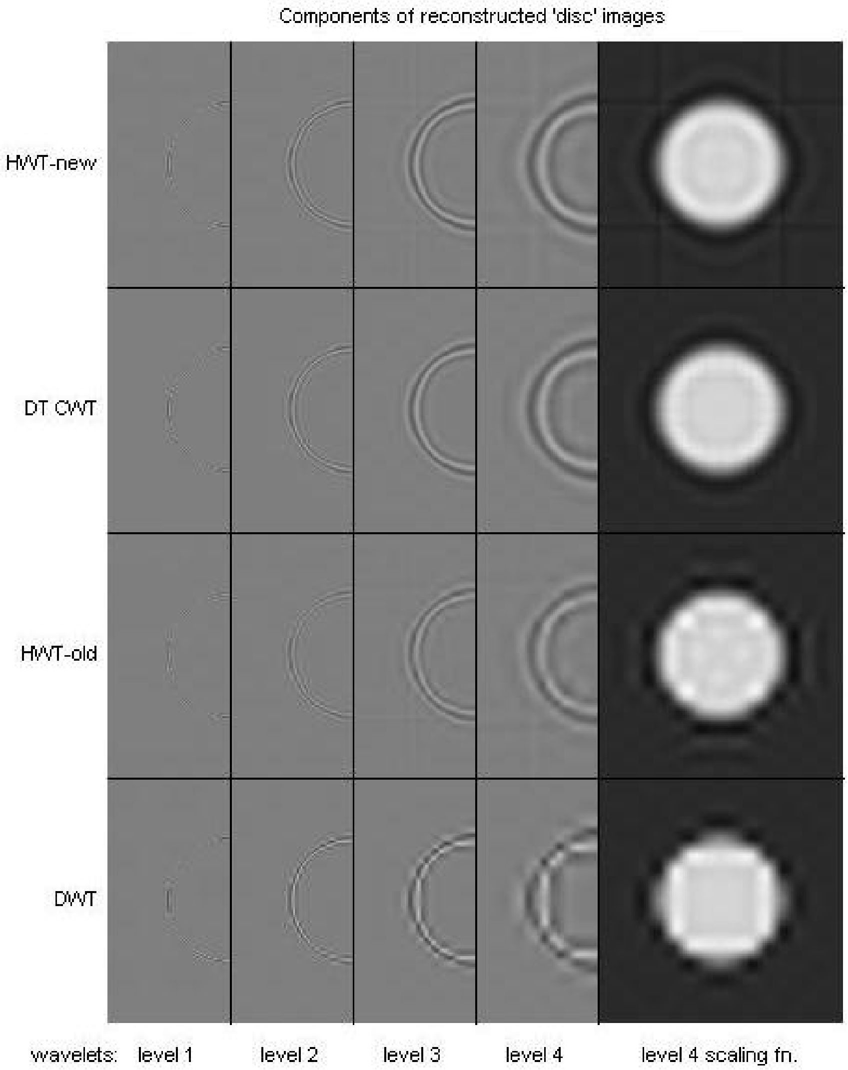


Figure 4. Comparison in the 2D case between the HWT, the DT CWT and the DWT.

It is obvious that our transform, in its new implementation, approaches Kingsbury's DT CWT. For the simulations we have used 10-taps Daubechies for HWT, and for the DT CWT and DWT we have kept the mother wavelets used in [3].

As the HWT is a complex wavelet transform, it has all the benefits of these types of transforms, including the directional selectivity, a very useful property when referring to denoising applications. Directional selectivity is a result of HWT's computation algorithm [5], thereby it is not affected by the IHWT implementation, this being the reason for which we do not insist on it in the present paper.

## IV. CONCLUSION

HWT is a very modern WT, being formalized only a few years ago [10]. In this paper we have presented an improved implementation scheme for the IHWT, which leads to better results from the shift-invariance point of view.

If ideal Hilbert transformers had be used, the two IHWT computation schemes presented in figures 1 and 2 would have lead to identical results. Unfortunately, the ideal Hilbert transformers are impossible to be exactly implemented. As the Hilbert transformer already implemented in Matlab did not provide a good reversibility, we have implemented a new Hilbert transform, based on the FFT algorithm, which provides better results in this direction. In the present paper we have illustrated through tests the superiority of the scheme relying on the reversal of the Hilbert transform associated to the Hilbert transformer newly implemented compared to the old implementation scheme using the Matlab implementation of the Hilbert transformer.

The reason for the two tests presented is to emphasize the contribution of every decomposition level to the translation invariance. The procedure at the basis of these tests follows three steps: the computation of the forward transform using four levels of decomposition, the cancelling of all the coefficients that do not correspond to the selected decomposition level and the computation of the reverse wavelet transform. The quality of the results obtained by applying both tests can be appreciated visually. To facilitate the first test results' interpretation, an objective criterion was also proposed. The results of its application are presented in table 1.

The main advantages of the proposed IHWT implementation over the other existing ones is the simplicity, the flexible structure, as we can use any orthogonal or biorthogonal mother wavelets, and the reduced computation time required.

In the future we intend to implement this transform in image denoising algorithms and we expect better results due to the increased shift-invariance. Preliminary tests prove this affirmation.

Also, we intend to study the dependence of the denoising result to the type of mother wavelet chosen.


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